PRINCIPLES AND METHODS OF TEACHING ARITHMETIC

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PRINCIPLES AND METHODS OF TEACHING ARITHMETIC

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PREFACE

This book, as its title indicates, is a discussion of principles and methods of teaching arithmetic. A complete study of methods of teaching any subject involves two things (a) A study of guiding principles, and (b) A study of the details of method—of possible methods of presenting the various parts of the subject.

The best place to study the details of method is in connection with a review of subject matter, but such a study should be preceded or accompanied by a study of general principles. It is the purpose of this book to do three things (a) To give a clear conception of the ends to be accomplished through the work in arithmetic (b) To analyze the teaching of arithmetic into the different kinds or types of teaching that occur in the subject and to give an understanding of guiding principles and a knowledge of possible methods of procedure, tools and devices to be used in each of these types; and (c) To make clear, by means of numerous illustrations and lesson plans, how these general principles and methods of procedure apply to the teaching of the particular subject of arithmetic.

The material of this book has all been developed and used in the author's classes during the last six years, and is presented here in book form in the hope that it may contribute something toward improving and standardizing the teaching of arithmetic. It is believed that this book will be found suitable for use as a text for normal schools and teachers' reading circles.

The author wishes to acknowledge his indebtedness to the books referred to in the text and bibliographies, to

the critic teachers of the Bowling Green State Normal College for the assistance they gave in testing out some of the suggestions made in the text and for the preparation of lesson plans, and to his wife for her assistance in the preparation of the manuscript and in correcting the proof.

JAMES ROBERT OVERMAN.

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PRINCIPLES AND METHODS OF TEACHING ARITHMETIC

PART I

INTRODUCTION

CHAPTER I

THE ENDS TO BE ACCOMPLISHED THROUGH THE TEACHING OF ARITHMETIC

The Importance of a Definite and Correct Conception of the Ends

Successful teaching of arithmetic is impossible unless the teacher and supervisor have a correct and clearly formulated conception of the ends they are trying to realize through their teaching of the subject. This truth is so self-evident as to be axiomatic, but in spite of this fact many teachers of the subject have only a vague idea of what they are trying to do. Perhaps the majority of teachers have no object in view except to so teach that their pupils will be able to work the problems in the text from day to day and to pass examinations. Indefinite and mistaken ideas of the purpose of the work in arithmetic are responsible for more poor teaching of the subject than any other one thing.

Waste. In the first place, without a definite conception of the ends to be attained, a teacher is not apt to accomplish much of anything, certainly not what she should accomplish. Purposeless teaching, or purposeless work of any kind, is always exceedingly wasteful. In order to

obtain the maximum of results with a minimum of effort, the teacher must know exactly what results are wanted.

What and How to Teach. In the second place, every thoughtful teacher of arithmetic is constantly confronted with two types of questions—what to teach and how to teach it. It is impossible to decide upon the relative importance of different topics and problems, or upon the relative merits of different methods, in any way except in terms of the ends to be accomplished. No one topic is any more important than any other topic, in and of itself; no one method is intrinsically better than any other method. The most important topics and the best methods are those that will most nearly accomplish the ends sought.

Measurement of Results. Finally, a clear conception of the ends that should be accomplished is necessary to enable the teacher to judge intelligently of the success of her work. The degree of approximation of the results actually obtained to the ends desired is the measure of teaching efficiency. The teacher who has only a vague idea of the aim of her instruction can form no adequate judgment as to its success, while the teacher who has a wrong conception of aims is very apt to form a wrong estimate of her work—to regard as entirely successful, work that is seen to be almost a total failure when judged in the light of the proper aims.

To summarize, it is essential that every teacher have a definite, correct, and usable idea of the ends sought through the instruction in arithmetic in order to (*a*) avoid waste, (*b*) intelligently select subject matter and method, and (*c*) have a standard by which to judge results.

The Scientific End

The Arithmetic of the Middle Ages. In order to see how the ends determine both the content and method of instruction, one needs to study the history of the teaching of

arithmetic. During the middle ages in Europe, the learning was entirely in the hands of the priests. There were few schools except those conducted by the priests in monasteries.

The course of study in these schools was usually confined to the trivium and quadrivium. The former comprised the three arts of grammar, logic, and rhetoric, but practically meant the art of reading and writing Latin; the latter included arithmetic, geometry, music and astronomy. The trivium and the quadrivium together constituted the seven liberal arts. Any student who studied beyond the trivium was looked upon as a man of great erudition. Two things are of importance in this connection, first, arithmetic was taught only to adults, and second, it was taught only to advanced scholars and men of learning. These facts have had a great influence upon the teaching of the subject down to the present time. The men who taught and studied arithmetic in these schools were not interested in practical affairs and, as a result, they did not study the practical side of the subject but studied it simply as a science—the science of numbers.

The arithmetic taught in the monastic schools was taken from the Greeks, the text used being Boethius' Latin translation of the arithmetic of Nichomachus, a Jew who wrote under Greek influence. This book, besides discussing fractions, ratio, proportion and progressions, devoted much space to a consideration of different kinds of numbers. In addition to the classes that we know today, such as odd, even, prime and composite, there were many others. Among these were perfect, redundant, defective, triangular, square, pentagonal, pyramidal and cubical numbers.*

*A perfect number is one that equals the sum of all its divisors except itself: thus, $6=1+2+3$, $28=1+2+4+7+14$. A redundant number is one the sum of whose divisors exceeds the number itself; thus the sum of the divisors of 18, $1+2+3+6+9$, is greater than 18, and

When arithmetic was handed down to the elementary schools, it was dominated by the scientific, scholastic attitude of the middle ages. The chief aim was to teach the pupils the science of numbers. The subject matter was not practical and the methods and organization were those suited to mature scholars. The influence and results of this point of view are felt to the present day.

Effect on Topics Taught. As the sole object was to impart to the students as much information as possible about the science of numbers, anything and everything that had to do with numbers was taught. As a result we find in early American arithmetics many topics that never were practical and many others that had long ceased to have any practical significance. Our early arithmetics contained such topics as G.C.D. by the Euclidean method, L.C.M., progressions, complex fractions, longitude and time, annual interest, exchange, chain rule, position, repeating and circulating decimals, compound, distributive, and conjoined proportion, permutations and combinations, duodecimals, powers and roots, medial and alternate alligation, barter, rule of false position, single and double rule of three, partnership, partial payments, tonnage of vessels, general average, equation of payments, bankruptcy, present worth, true discount, practice and gauging. In short, they were compendiums of everything that had to do with numbers—their theory and application. The prac-

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the sum of the divisors of 12, 1+2+3+4+6, is greater than 12. A defective number is one the sum of whose divisors is less than the number; thus the sum of the divisors of 8, 1+2+4, is less than 8, and the sum of the divisors of 16, 1+2+4+8, is less than 16. Triangular numbers are 3= , 6= , 10= , etc. Square numbers are 4= , 9= , etc. Four is the smallest pyramidal number because four shot can be piled in the form of a pyramid, three on the bottom to form a triangular base and one on top. Five and ten are other pyramidal numbers.

tical value of a topic was never considered, if it concerned numbers it belonged in arithmetic. Many of the above topics have disappeared from our arithmetics and even their names are unknown to the present generation, but others, unfortunately, are still present—in our arithmetics today simply because they were put in when arithmetic was everything pertaining to number and have been there ever since.

Effect on Problems. As long as arithmetic was regarded as the science of numbers, the only requisite of a good problem was that it involve the desired number relations. So under the dominance of the scientific aim, the problems in the arithmetics were largely of the puzzle type. The following are typical examples taken from an old American arithmetic published in 1859.

1. A man driving his geese to market, was met by another who said, "Good-morning with your hundred geese." He replied, "I have not a hundred; but if I had half as many more than I have, and two geese and a half, I should have a hundred." How many had he?

2. The head of a fish was 9 inches long; its tail was as long as its head and half its body, and its body was as long as its head and tail both. What was the whole length of the fish?

Effect on Methods. The methods used in early American schools were largely those handed down from the middle ages when the subject was taught to adults and advanced scholars. No attempt was made to adapt the subject matter or methods to the needs and abilities of the children, or to appeal to their interests and make the subject interesting. Arithmetic was taught simply as a body of facts and rules about numbers. These facts and rules were not usually proved or justified in any way but were given arbitrarily. The pupils first learned the rule and then solved a list of problems based on that rule.

Effect on Organization. We have seen the effect of the scientific end on the methods of teaching and on the content of arithmetic—the topics taught and the type of problems. It had just as marked an effect in another way in that it led to a logical organization of the subject matter—an organization according to the subject matter itself rather than according to the needs and abilities of the learner. This was but natural as it was worked out by mature minds for mature minds and the center of interest was the subject itself and not the learner. As a result the organization was topical. Texts usually started with a chapter on Notation and Numeration, in which the reading and writing of numbers was rather fully treated. This was followed, in order, by chapters on Addition, Subtraction, Multiplication, Division, Properties of Numbers (L.C.M., G.C.D., Tests of Divisibility, etc.), Common Fractions, Decimals, Denominate Numbers, Ratio and Proportion, Percentage, Application of Percentage, and Mensuration—each topic being fully treated before taking up the next.

The Disciplinary End

Definition. Since the scientific end led to the teaching of many topics in arithmetic that had no practical value, it was but natural, as the general public became better educated and better acquainted with the work of our schools, that complaints should arise. This is exactly what happened, and in order to defend the teaching of so much useless and obsolete material, educators called to their aid the psychological doctrine of formal discipline. This doctrine held that a general power or ability may be acquired through the study of one subject and that once acquired, it will operate in full in any other subject or situation no matter how different the new situation may be from that in which the power was originally developed.

The analogy was made to physical development. One may develop the muscles of the arm by swinging dumb bells, but after the muscles have been thus strengthened the increased strength can be used in driving nails or any other act in which the muscles of the arm are used. Under such a doctrine, it would make very little difference what is taught just so it is not too easy. Educators found in this doctrine their defense of the useless material that had accumulated in our arithmetics under the influence of the scientific end. Such material might not be of any practical use, they admitted, but it served to develop and train the mind.

Effect on Topics. The aim of formal discipline did not change either the topics or the character of the problems from what they had been under the scientific end. Its effect was simply to defend and perpetuate all of the useless and obsolete material that was then in our arithmetics. Some of this material, such as G.C.D. by division (Euclidean method), cube root, etc., never had any practical value for our students; other topics, such as Partnership and Partial Payments, had once been practical but had ceased to be so because of changes in business procedure.

Effect on Problems. Since the aim was simply to develop the mind, the intricate puzzle type of problem was still regarded as better than the type of problem that really occurs in life, which is usually comparatively simple. This attitude is still quite common. The author recently heard a teacher of many years' experience defend the use of puzzle problems on the ground that life problems are too easy.

Effect on Methods. Another result of the disciplinary end in our schools was to place greater emphasis upon analysis and explanation than upon ability to do. As much stress was laid on the ability to explain carrying in addition and borrowing in subtraction as upon a

mechanical mastery of these processes; the ability to give long, formal analyses of puzzle problems was considered to be as important as the ability to solve life problems.

The effects of the scientific and disciplinary ends are still felt today in the organization of our work and still more in our methods of teaching. The author recently visited a fourth grade in one of our large cities, and found the pupils analyzing examples in long division. The pupils were slow and inaccurate in carrying out the process but they could give beautiful analyses—all alike.

The Social End

Origin. It has been shown how the scientific end originated with the Greeks and was handed down to us by the monastic schools of the middle ages, and how the doctrine of formal discipline was later brought to its defense. Although the influence of these ends has been very marked and has persisted to the present time, another end or point of view has existed parallel to these for several centuries and at the present time is gradually superceding them. This point of view had its beginnings, as far as Europe is concerned, at the time of the Rennaissance. The Rennaissance in arithmetic was brought about by the gradual introduction of the Hindu notation, knowledge of which came to Europe from the Arabs of Spain and northern Africa. This gave the people of Europe a practical system of notation and of calculation and enabled them to develop the practical side of arithmetic. The Greeks had already done something along this line but were handicapped by the lack of a usable system of notation. The people of Europe, particularly the Italians, now took what little they had received from the Greeks of a practical nature, combined it with the arithmetic of the Hindus and with this as a basis, developed the practical side of the subject as we know it today.

Our early American arithmetics drew their subject matter from both these sources and as a result were a mixture of the science of numbers inherited from the Greeks and Romans, and the art of calculation and its practical applications inherited from the Hindus and Europeans. Thus we find in these books both the Greek proportion and the Hindu rule of three, which was simply proportion in a different guise and under a different name. We also find both duodecimal fractions, which originated among the Romans, and decimal fractions which probably originated in Germany during the sixteenth century.

Definition. The doctrine of formal discipline or mental training, set up by educators as a defense for all the useless material included in our early American arithmetics, satisfied the people for a while, but gradually the protests of the business man and the practical man in all lines of work became too strong to be met with this doctrine. Educators today have come to realize that it is the business of the schools not to teach the science of arithmetic as such to the pupils, but rather through the teaching of arithmetic to prepare the pupils for life. This change has been hastened by the overthrow of the doctrine of formal discipline on psychological grounds. This new aim of preparation for life may be called the *Social Aim*. It is so important that it will be discussed in detail in the next chapter.

CHAPTER II

THE SOCIAL ENDS IN ARITHMETIC

It is very well to say that today the aim in teaching arithmetic is to prepare the pupils for life, develop socially efficient individuals, etc.; but to stop with such general statements is to have nothing but high-sounding, meaningless words. Such a statement of ends is too general, too indefinite to be of much assistance as a guide in teaching. To be of the greatest service, the formulation of ends must state in definite terms what can and must be done in teaching arithmetic in order to prepare the pupils for life.

I. Automatic Mastery of the Fundamentals

In the first place, if the pupils go into any line of work in which they need to use arithmetic at all, they must know a few simple facts and be able to carry out simple calculations with accuracy and reasonable speed. Practical men today are constantly complaining that the graduates of our common schools cannot add, subtract, multiply and divide, and that they are hopelessly lost in the simplest example in fractions. A mastery of the fundamentals, then, is the first thing necessary for adequate preparation for life. Further, this mastery must be of a particular kind. One might have a mastery of cube root, for example, in the sense that if he wanted to find the cube root of a number, knowing the algebraic theory, he could think out the process step by step. Such a mastery might be called a logical mastery. This is not what is needed in the fundamentals of arithmetic. It is not enough that one be able to think out the process of long

division; one must be able to perform it accurately and with reasonable speed *without thinking.* In short, the mastery must be mechanical or automatic rather than logical. So, if our pupils are to be adequately prepared for life, we must constantly keep in mind that—*The first social aim of instruction in arithmetic is to give the pupils a mechanical, automatic mastery of the fundamental facts and processes. This mastery must be both accurate and speedy; it must be permanent, not temporary.*

The Fundamental Facts and Processes. In stating the first aim, the term "fundamental facts and processes" was used. These are:

(*a*) The four processes of addition, subtraction, multiplication and division of integers, common fractions, mixed numbers, decimals and compound numbers of not more than two or three denominations.

(*b*) Common units of measure and their relations. (Tables of denominate numbers.)

(*c*) The fundamentals of percentage. There are seven of these: changing per cents to common and decimal fractions; changing common and decimal fractions to per cents; finding per cents of numbers; finding what per cent one number is of another; and finding a number when any per cent of it is known.

(*d*) The rules of simple mensuration.

(*e*) Ratio and proportion. Pupils should have a knowledge of the meaning of these terms and enough practice to enable them to find the ratio of any two numbers, and to form and solve a proportion.

(*f*) The use of simple mathematical formulas and tables.

The above constitute the "tools" of arithmetic. All else is but application of one or more of these. Without a thorough mastery of the tools, no successful work in arithmetic is possible.

II. Power to Apply Fundamentals to Concrete Situations

A mere mastery of the tools of arithmetic, however, is not sufficient. One must know when and under what circumstances to use these tools. It is not enough that our pupils know how to add and divide, they must also know when to use these operations; in other words, they must be able to "size up" an arithmetical situation and select the tools necessary to meet the situation successfully. Further, it is not sufficient that pupils be able to use their arithmetical tools in meeting arithmetical situations in a book—they must be able to use them in meeting situations outside of books. So to our statement of aims we must add the following:

The second social aim of instruction in arithmetic is to develop in the pupils the ability to grasp, interpret, and master the simple arithmetical situations that are of common occurrence in life.

III. Ability to Think

If the pupils are to be adequately prepared for life, the schools must give them something more than the mere mastery of the subjects taught. They must prepare them to meet new and unforeseen situations; knowledge and habit, alone, will not enable the pupils to do this. Knowledge is necessary but not in itself sufficient. New situations can be successfully met only by exercise of the higher powers of thinking and reasoning. Can the schools do anything to develop these powers? Undoubtedly, yes. They can and they do. Pupils get more or less "mental discipline" from every subject studied. Arithmetic is no exception, in fact it affords unusual opportunities for such discipline, better than afforded by any other elementary school subject except history and geography.

Admitting the possibility of mental training is an entirely different thing, however, from justifying the teaching of any subject on disciplinary grounds alone. Since this discipline can be obtained from any subject, properly taught, disciplinary value is not in itself enough to justify the inclusion of any subject or topic in the curriculum. If disciplinary value exists, whether it be great or small, it is a by-product to be gained more through the method of teaching than through the particular subject taught. Arithmetic is justified in the elementary school curriculum solely because of the practical value of its subject matter. This does not mean, however, that we should not do all in our power to so utilize this practical material as to develop the child in every possible way.

The disciplinary value of arithmetic is largely the training in logical thinking that can be given by the subject if properly taught. Therefore, if the schools are to do all possible through arithmetic to prepare for life, a third aim must be set up, as follows:

The third social aim of instruction in arithmetic is to so teach that the pupils develop their inborn power to think, form the habit of thinking things out for themselves and of verifying their conclusions, and form a just estimate of the usefulness of thinking.

Effects of the Social Ends

Effect on Topics. We have seen that American arithmetics inherited a vast wealth of material concerning numbers from two separate sources—the Greeks and the Hindus. All of this material was included in our early arithmetics and justified because it had to do with numbers and was supposed to develop the mind. In order to realize the social aims of arithmetic, however, we do not need all this material but must select that which will help us accomplish our ends and discard the rest.

To accomplish the social ends we do not need to teach anything except those topics that will help prepare the pupils for successful and useful participation in life. The sole test of what to teach must be—Is this fact or process one that the pupils may be called upon to use in life, one that may help some of them to succeed?

Eliminations. If we set up such a standard as the above, it will mean that many topics now in our arithmetics must be eliminated. The following are examples:

1. Greatest common divisor.
2. Least common multiple as a separate topic.
3. Fractions with large denominators. The common fractions taught should be those actually used in business and in practical life.
4. Obsolete tables and units in denominate numbers and those used only by specialists, as, Troy and Apothecaries' Weight, Surveyors' Measure, long ton, hand, rood, furlong, etc.
5. All work with compound numbers of more than two or three denominations.
6. All problems in Longitude and Time.
7. Compound Proportion.
8. The long process of extracting cube root. Even square root is doubtful. It would be better to substitute tables of square and cube root and teach the pupils how to use them.
9. Complex Fractions.
10. All applications of percentage that do not conform with present day business practice. Among these are: true discount, partnership, partial payments, equation of payments, foreign exchange and annual interest.

Additions. It has been seen that some topics in the older arithmetics are being eliminated because they are not socially important. At the same time, the course is being

enriched by the addition of other topics that are socially important and that were not in the older books. Among the most important of these are the following:

1. Algebra. At one time it was the custom in many of our schools to give a short course in algebra in the seventh or eighth grade. This work was usually of the most formal, uninteresting and useless kind, and as a result prejudiced many people against algebra below the high school. At present an entirely different sort of algebra is being taught in many of our schools. It is not presented as a separate subject but grows naturally out of the work in arithmetic, particularly in mensuration and percentage. Literal notation, the equation and the formula are taught as tools to be used in the solution of arithmetical situations.

2. Geometry. In addition to the usual work in simple mensuration, many schools at the present time give some time to simple surveying (based on the right triangle, and on similar triangles), and to the applications of some of the more important facts of geometry to practical situations such as carpentry and shop work. The geometric facts used are not proved but are discovered by experiment.

3. Mathematical tables such as square root, cube root, simple and compound interest tables.

4. Statistical Graphs. The use of graphs to present statistical data is so common in newspapers, magazines and books that it is the duty of the schools to teach their pupils how to make and interpret such graphs. In many schools the pupils are given practice in constructing graphs showing the variations in the temperature of the school room for one school day, the absences for one month, their scholarship records in the various subjects, etc. Practice is also given in interpreting graphs in current newspapers and magazines.

5. A more careful and detailed study is being made in

the upper grades of the social situations underlying the more important and fundamental business applications of arithmetic, such as Banking, Taxation, Insurance, Methods of Investing Money, etc.

Effect on Problems. The accomplishment of the social ends demands life rather than puzzle problems. If we are to prepare our pupils to solve the arithmetical situations that they will meet in life, we must give them life problems in school. So we find our newer arithmetics giving problems that actually occur in business, farming, cooking, sewing, carpentry, shop work, manufacturing, building a house, furnishing a home, etc. The traditional text book problem is disappearing and life problems are taking their place.

Effect on Organization. Under the influence of the social aim, the center of attention is shifted from the subject itself to the learner. The modern school is no longer content with teaching the subject to the child but rather is attempting to teach the child through the subject. This changed point of view has led to a radical change in the organization of the subject matter. The subject itself has given place to the learner as the basis of the organization. The material, instead of being grouped under the large subdivisions of arithmetic is presented in the order that seems to be best adapted to the abilities and interests of the pupils. Those topics or parts of topics that are most needed at the time, and which are within the grasp of the child, are given first, and as the needs and abilities of the children develop other topics and parts of topics are added. Such an organization is called psychological, and it differs radically from the old logical order. Instead of completing each topic before going on to the next, the easiest parts of each and those most needed at the time are taken up first and the more difficult phases are postponed until later. This leads to a spiral arrangement in which each

topic recurs several times in the course, but with something new added each time. As an example, under the old logical organization, the four operations with integers were completed before taking up the work in common fractions. The pupils, however, need to know something about such simple fractions as ½, ⅓ and ¼ before they need to know anything about carrying in addition and borrowing in subtraction. So, under a psychological organization, the work with fractions is started long before the work with integers is completed and recurs at intervals throughout the course—the work already done being reviewed and more added at each recurrence.

Effect on Methods. Under the influence of the scientific and disciplinary aims, it has been seen that much emphasis was placed on explanations and analyses of processes and of problems. Under the social aim it does not matter whether the pupils can explain a process or not, providing they can perform it with accuracy and speed. Social efficiency does not demand the ability to explain "borrowing" in subtraction or the process of long division, but it does demand a mechanical mastery of these processes. This should not be taken to mean, however, that these processes should not be justified to the pupils when first presented. In the case of problems, the social emphasis is on the ability to solve rather than on the ability to give long formal analyses or explanations. Whatever analysis of problems is done in the schools should be solely for the purpose of helping the pupils to solve other problems.

Many other changes of method have come from the shifting of the center of attention from the subject to the child. These will be treated in detail throughout the book. Among the most important of these changes are the changed attitude toward objective work, the use of games, stories, and dramatization, the change in the character of

the drill, the attempt to motivate the work, the utilization of the pupil's own interests and activities, the setting of concrete standards, and the measuring of results.

Present Status

It would be a mistake to suppose that all these changes, due to change of aim, have already occurred and are finally accomplished. This is not the case, the present is a time of transition. Although in theory, educators have abandoned the scientific and disciplinary aims, in practice they still dominate much of our work. Many of the text books now in use were written under the influence of these aims, and even the best of our books show their influence to some extent. At the present time, however, American arithmetics are making rapid strides towards complete socialization. The author recently asked a teacher of many years' experience to examine a set of arithmetics just published. She returned them with the comment, "Why, these are not arithmetics, they are books on sociology."

The case is not so hopeful with teachers and methods as it is with text books. Many of the methods of teaching, in common use in our schools today, go back to the old ends and are poorly adapted to the work of socializing the pupils. The greatest obstacle in the way of bettering the teaching of arithmetic is the teacher. Too often she is entirely lacking in knowledge of or sympathy with the social point of view. Her sole object is to "teach the book." Teachers and supervisors must be thoroughly imbued with the social point of view before better teaching of arithmetic is an accomplished fact in our schools.

CHAPTER III

THE COURSE OF STUDY

Time of Starting the Formal Study of Arithmetic

Formerly in this country arithmetic as a formal study was begun in the first grade and continued throughout the elementary school. Recently there has been a tendency to postpone the formal teaching of the subject until the second or, in some cases, the third school year. Where this is done, sometimes no arithmetic at all is taught in the first grade but usually the subject is taught incidentally in connection with the other work of the grade.

Should Arithmetic be Postponed? Some of the reasons given for postponing the subject are (1) Arithmetic is too difficult, the average first grade pupil is not mature enough to begin the formal study of the subject. (2) Its study in the first grade is unnatural and forced; the pupils have no real need for the subject. (3) It is uninteresting to the pupils. (4) It calls for reasoning of which the pupils are incapable. (5) It is not necessary to teach arithmetic in the first grade. All of the subject that should be taught can be mastered in six years or less. (6) The difficulties of arithmetic should be postponed until after the mechanics of reading have been mastered. Starting both subjects at the same time is too much of a burden for the ordinary school beginner.

Is Arithmetic too Difficult? The belief that arithmetic is too difficult for the average first grade pupil originated at a time when the subject matter of arithmetic was complex and unsuited to the child and the methods of teaching were abstract and formal. Such a treatment of arithmetic

was undoubtedly too difficult and if it were the only kind possible, schools would be justified in postponing the subject until at least the third grade. But with material carefully chosen according to the comprehension and interests of primary children and with concrete, interesting methods of instruction, the situation is entirely different. Beyond question, beginning school children are mature enough for the study of any arithmetic that a modern course of study would assign to the first grade.

Is Arithmetic Needed? It would be entirely possible for pupils to get along without any formal study of arithmetic in the first year. The need is not pressing but it exists. The first grade pupils make as much use of their number work as of their reading or anything else that they study in school. Their games involve counting in choosing sides and keeping score; they are constantly encountering number ideas and facts in their other school subjects and in their simple purchases and other activities outside of school. That this need exists is shown by the fact that almost all pupils have some knowledge of numbers before entering school and it is the duty of the school to satisfy and expand this need as rapidly as possible.

Is Arithmetic Uninteresting? The belief that arithmetic is uninteresting to the child in the first school year is, again, due to the wrong kind of arithmetic. With material and methods suited to the children, the subject becomes one of the most interesting in the curriculum. All children naturally like to count, the rhythm makes an instinctive appeal; and with a proper use of objective work, construction work and games, the whole subject is far removed from drudgery and can be made a real pleasure.

Is the Reasoning of Arithmetic too Difficult? All the evidence available indicates that first grade pupils can think and reason quite as well as memorize, providing always that the situations in which they are asked to reason are

suited to their degree of maturity and come within the range of their experience. What first grade teacher has not been frequently astounded by the power of her pupils to reason and draw logical conclusions?

Is Six Years Enough? Those who claim that it is not necessary to start teaching arithmetic in the first grade, because it is possible to teach the essentials of the subject in five years, lose sight of two things. In the first place, the school life of many pupils is very short. For this reason the essentials of arithmetic and the most important applications of the subject should be completed as early as possible, by the end of the sixth school year at the latest. This is also desirable because of the present tendency to make the break between the elementary and the high school at the end of the sixth instead of the eighth school year as formerly.

In the second place, although it is possible to teach all the essentials of the subject in six or even in fewer years, it would not be advisable to do so. In order to assure the retention and mechanization of the number facts and processes, it is essential that the pupils' experience with them be as prolonged as possible. One may memorize a poem in an hour's time, but most people will soon forget it. If, however, the poem is relearned several times, over an interval of months or years, it will eventually be fixed permanently in mind. In order to give the pupils a *permanent* mastery of the fundamental facts and processes, the drill on them must be spread over as long a period of time as possible.

Is Reading Enough? The argument that the pupils should not be required to start the study of two such difficult subjects as reading and arithmetic at the same time, is probably the strongest argument against starting the study of arithmetic in the first year. The experience of many schools, however, would seem to indicate that the

average class is perfectly capable of beginning both subjects at the same time, and that if the first school year is devoted to reading alone it becomes very monotonous and uninteresting. With a weak class or in communities where the pupils come from the homes of foreigners, the difficulties of mastering the mechanics of reading may be sufficient to justify the postponing of arithmetic until the second year.

The Present Practice. In a report on Mathematics in the Elementary Schools of the United States, made in 1911 as part of the American Report to the International Commission on the Teaching of Mathematics, a summary is given of answers received by the committee to a questionnaire, sent to school superintendents, asking the year in which the study of arithmetic was commenced in their schools. Of the ninety replies received, 71½ per cent stated that arithmetical study was begun in the first year, 22 per cent in the second and 6½ per cent in the third.

Should There be a Definite Requirement in the First Grade? Some schools that teach arithmetic in the first grade specify a definite amount of work to be accomplished and assign a certain amount of time to the study of the subject, while others teach the subject incidentally, teaching only such number work as arises in connection with the other subjects. Experience everywhere has shown that incidental number work means no number work—unless the teacher is held responsible for accomplishing a certain amount of work and a definite amount of time is devoted to it, nothing is accomplished. This does not mean that the method should not be incidental—to a large extent it should be; that is, the number work should be made to grow naturally out of the other school work and should be correlated with it. The teacher, however, should know exactly what number work she is expected to make grow out of the other work and further should have a definite

time allotted in which to accomplish this and fix the work in the pupils' minds.

Conclusions

A definite amount of number work should be required of the first school year and a definite allotment of time should be made to the subject for the following reasons:

1. Incidental teaching accomplishes nothing definite.
2. Arithmetic needs to be started in the first grade, in order to provide the necessary number imagery and afford time for the fixing of number concepts before starting on the mechanical manipulation of symbols, and
3. To provide for spreading the drill on the fundamentals over a long period of years in order to insure retention, and
4. To make it possible to separate difficulties, and
5. So that the fundamentals and the most important applications can be completed by the end of the sixth school year. This is necessary (*a*) because of the large number of pupils leaving school at the end of the sixth year, and (*b*) because of the movement towards a six year elementary school.
6. Children's interests, activities and needs afford opportunities for and demand some arithmetic.
7. Pupils in the first year are capable of doing any work in arithmetic that a modern course of study would require.

Summary of Work in United States and Foreign Countries*

First School Year

United States. In many of the schools of the United States only incidental number work is given during the

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first school year. In some schools regular number work is begun the latter half of the first year. In a few schools incidental number work is given during the first two school years.

A great variety of courses exists in those schools which provide special periods for number work during all or a part of the first school year. The following outline is from the New York State course of study. It respresents one of the most advanced courses.

Pupils are taught to count, read, and write numbers to 100 and to memorize the 45 addition combinations. The drill in these combinations is given in such a way as to prepare for subtraction as well as addition. Pupils learn to count 100 by twos, fives, and tens. The children are taught to carry in addition. No attempt is made to teach the science of numbers; the art of computation is emphasized. Oral work greatly predominates, but a good deal of seat and blackboard work is given.

General Summary of the First Year's Work. There is not much divergence between the courses in arithmetic in the most progressive schools of the various countries. In general it may be said that the aim is to teach the children to count and to read and write the numbers to 100; to perform easy additions and subtractions within these limits; to know the fractions ½, ⅓, and ¼; and to make a few easy multiplications and divisions involving numbers less than 20. Practically all the work is oral, and objects are freely used.

Second School Year

United States. In the New York State course there is continued drill on the use of the 45 combinations in addition and subtraction. There is also drill on series in addition and counting by twos, threes, fours, and fives. The addition method is used in subtraction. There is con-

tinued drill in rapid additions. The pupils memorize the 45 combinations in multiplication. These are so taught as to prepare for division at the same time. The process of carrying in multiplication is taught. Good model forms are extensively used. No explanation of the processes is attempted.

General Summary of the Second Year's Work. The course of the second school year varies more than that of the first year. In general, the aim of the work may be said to be to teach the children to count, read, and write numbers to 1,000; to perform the fundamental operations on numbers less than 100; and to learn the simple units of measure. In several countries, multipliers and divisors are limited to one figure. The pupils are taught to count to 100 by twos, fives and tens. The 45 addition combinations are learned in this year and the multiplication tables involving products up to 10 times 10 are usually studied.

Subtraction is usually taught by the addition method and is studied at the same time as addition. The fractions ½, ⅓, ¼ are taught, and objects are very extensively used. The simple denominate numbers are studied and much attention is devoted to measures and estimates. Oral work predominates. Numerous concrete problems involving the experiences of the pupils are given.

The course in the most progressive schools of the United States compares favorably with the most advanced courses in Europe. In the great majority of the schools of the United States, however, not so much is attempted in arithmetic as in the best schools of Europe during the second school year. The longer school year and the longer school day enable the European teachers to devote more time to drill in fixing the number facts, and the pupils leaving the second grade there are probably more thoroughly grounded in the fundamentals than is the case in this country.

Third School Year

United States. (New York State course.) Drill in counting is continued. The pupils are taught to count by fives to 100, beginning with 0, 1, 2, 3, or 4, and by sixes, beginning with each of the numbers from 0 to 5, inclusive. In short division, 2, 3, 4, 5, 6, 7, 8, and 9 are used as divisors. Multiplication, with two or more figures in the multiplier, is taught. Definitions of the terms addend, sum, minuend, subtrahend, remainder, multiplicand, multiplier, product, dividend, divisor, and quotient are learned. Pupils are taught to measure, using the inch and the foot. Square inch and square foot are also taught. The fractions ½ and ¼ are applied to the use of the linear unit in measuring.

During the second half of the year the following topics are taught: long division, multiplication tables of the tens, elevens, and twelves, and their use as divisors in short division; tests for divisibility by 2, 3, 5, 9, and 10; the definition of factor and prime factor. The pupils memorize the prime factors up to 25; linear and square measurement of objects in the school room, and liquid measure are taught.

A great deal of attention is given to oral drill and written work for accuracy and rapidity in the four operations. At the close of the year the pupil is expected to be able to add, subtract, multiply, and divide integers with accuracy and facility.

General Summary of the Third Year's Work. There is greater variety in the third-year courses than in those of the first and second school years. In a few of the countries—for example, Belgium and Italy—the notation of decimal fractions is introduced. This is usually not done in the United States until the latter part of the fourth or the early part of the fifth year. It is a common

practice abroad to introduce fractions with denominate numbers. In all of the European countries and in Japan oral arithmetic greatly predominates. In Japan a special part of the recitation is set apart for this oral drill. The text books in several of the countries, notably Germany, Austria, and Italy, are collections of problems rather than expositions of number.

Fourth School Year

United States. (New York State course.) Roman numerals are taught from 1 to 100 and by hundreds to 1,000. The pupils learn to read and write United States money; to use cancellation, when possible, in the solution of problems; and to use the terms, pint, quart, peck, and bushel. Common fractions are developed objectively. The pupils are taught to change fractions to equivalent fractions of higher and lower denominations; to add and subtract fractions the denominators of which do not contain more than two digits; to multiply a fraction by an integer and by a fraction, and to multiply an integer by a fraction; to divide a fraction by an integer and by a fraction, and to divide an integer by a fraction. The principles for multiplying or dividing a fraction by the proper operation upon its numerator or denominator are taught; also the effect of multiplying both terms of a fraction by the same number or dividing both by the same number.

There is continued drill throughout the year on the four operations with integers. Addition and subtraction of mixed numbers are taught. The pupils learn how to factor and to find the least common multiple of numbers to 100. Problems are carefully stated before being solved. Cubic measure is taught. Volumes studied include cubic inches, cubic feet, and cubic yards. Simple problems are given in bills and accounts.

General Summary of the Fourth Year's Work. The attempt is made to fix the four operations for abstract and denominate numbers firmly in mind by the time the pupil has completed his fourth school year. In all of the countries a large amount of both oral and written drill is provided. Speed and accuracy in the fundamentals are watchwords everywhere. In addition, the concept of common fractions is much extended, and the decimal notation is introduced. In several of the countries, only addition and subtraction of common and decimal fractions are taught, but in others multiplication and division are also included in the course. When this is done, only the easy cases are usually considered.

The general use of the metric system in the countries of Continental Europe makes the introduction of decimals practicable at an earlier date than in the United States. Usually the subject of decimal fractions is closely correlated with the metric system.

In general, it may be said that the courses in European countries include all that is offered in the United States during the fourth school year and a good deal of the work of the fifth school year. The formal study of common or decimal fractions is seldom begun in the United States before the fifth school year.

In most of the European countries emphasis is put upon computation rather than upon reasoning during the first four school years.

Fifth School Year

United States. (New York course of study.) The pupils are taught how to read and write decimal fractions and to reduce common to decimal and decimal to common fractions. The four fundamental operations with decimal fractions are presented, and the common aliquot parts are studied. The tables for linear, square, and cubic measure

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are reviewed. Ascending and descending reductions involving the various tables of denominate numbers are taught. Square measure is applied to finding the area of squares, triangles, rectangles, and to problems of painting, papering, and plastering. Cubic measure is applied to finding the volume of rectangular solids, the capacity of bins and cisterns, and the cost of masonry. Many problems are given involving avoirdupois, dry and liquid measure, English money, time and circular measure. Numerous problems involving reductions are given. The value of the franc, pound, and mark in United States money is learned. Problems involving simple bills and accounts are given.

General Summary of the Work of the Fifth School Year. The work of the fifth school year in most European countries is decidedly more extensive than in most of the schools of the United States. It is unusual in this country for the general ideas of percentage to be introduced before the sixth school year. In several of the European countries this work is introduced during the fifth school year. Proportion is rarely introduced in this country in the fifth school year, but it is not uncommonly introduced in that year abroad. Tests of divisibility and prime factors are given much more attention in the European countries than here.

The general use of the metric system gives the European teacher an excellent field of application for decimal fractions, and the two topics are closely related.

Common fractions and denominate numbers are more closely related than is usually the case in the United States.

Probably the most marked difference between fifth-grade work here and abroad is the large amount of time and attention put upon the propædeutic study of geometry in the European countries. Very frequently this work is

given under the subject of drawing, and when this is the case the arithmetic and drawing are usually taught by the same teacher.

It may be said, in general, that the work of the fifth year in the European schools is considerably in advance of the work in the schools of the United States. The courses abroad include all that is included in the fifth-year courses in this country and a good deal that is not included here.

Sixth School Year

United States. (New York State course.) The subject of common fractions is reviewed, especial emphasis being placed upon the three problems:

1. To find a fractional part of a number.
2. To find what fractional part one number is of another.
3. Given a fractional part of a number and its relation to the whole, to find the whole.

Denominate numbers are reviewed, and drill is given on industrial problems demanding their use. The idea of percentage is introduced, and percentage is applied to profit and loss, trade and cash discount, commission, simple interest, and the making of promissory notes. Some problems are given in simple interest in which three of the elements, principal, rate, time, and interest are given to find the fourth. The simple equation is introduced and used in the solution of some of the problems.

In some schools of the country the course includes, in addition to the above, the keeping of simple accounts, the making out and receipting of bills, and some simple measurements. These measurements are usually made in connection with the study of denominate numbers.

General Summary of the Sixth Year's Work. The course in mathematics in practically all of the European countries is decidedly more advanced than in the United States.

The sixth school year is the last year of primary instruction in many of the countries.

The courses abroad include all that is given during the corresponding school year in the United States and also many subjects that are not included in the course in this country.

Another marked contrast to the work in the United States is found in the emphasis that is put upon the propædeutic study of geometry. The courses abroad, almost without exception, provide for the study of intuitive or observational geometry. The amount of time given to this work varies, but the general prevalence of such work is indicative of the importance attached to it. In several of the countries provision is made for the study of geometric drawing. The pupils learn to use the ruler, protractor, compasses, and triangle, and to make the simple geometric constructions. This work is closely correlated with the work in intuitive geometry and the classes are usually taught by the same teacher. In Germany the systematic and serious study of geometry begins with the sixth school year. In Germany and England easy loci problems are introduced.

Short methods and abbreviated processes receive more emphasis abroad than in this country. This is especially true in Austria, Belgium, Germany, and Hungary.

Alligation is taught in several of the countries; for example, in Germany and Russia. The subject is seldom taught in the United States.

The elementary ideas of algebra and of algebraic computation are introduced during the sixth school year in a few of the European countries; for example, in France, Russia, and Sweden.

The time devoted to the study of mathematics abroad is about the same, on the average, as in this country. In some of the countries the time devoted to mathematics is

somewhat in excess of that in the United States, if we consider the course in drawing as a part of the course in mathematics.

A Course of Study in Arithmetic

The following course of study is based on that in use in the Training School of the State Normal College at Bowling Green, Ohio.

Grade 1

Rote Counting. From 1 to 10 and from 10 to 1 by 1's, from 10 to 100 by 10's, and then by 1's. The purpose of this work is to fix the number sequence.

Rational Counting. To develop the idea of number as answering the question "How many?" the numbers from 1 to 10 are developed objectively. The mathematical symbol (1, 2, 3, 4, etc.) is connected with the objective situation, the number picture (⁘, ⁘, ⁘, ⁘, etc.), and with the spoken and the written word.

Reading and Writing Numbers. In Hindu-Arabic symbols to 100, in words to ten.

Measuring. The pupils make use of the following units in their construction work, in playing store, and in the other activities of the grade—inch, foot, yard, cent, nickel, dime, pint, quart, day, week, and month.

Addition. 1. The combinations whose sums do not exceed ten are developed and drilled upon. As a preparation for subtraction each fact is drilled on in four ways:

5	5	4	4
+4	+?	+5	+?
?	9	?	9

The combinations are also drilled upon in the horizontal form (5+4=?, 5+?=9, 4+5=? and 4+?=9). The terms "addition," and "plus," and the sign "+" are used.

2. In keeping scores for games the pupils add columns of three addends each, not involving addition by endings (see page 123), as

$$\begin{array}{r} 1 \\ 3 \\ 5 \\ \hline \end{array}$$

Subtraction. In the last half of the year subtraction as a separate process is introduced. The subtraction facts are developed and drilled upon with the corresponding addition facts and in the forms

$$\begin{array}{rrrr} 4 & 5 & 9 & 9 \\ +5 & +4 & -4 & -5 \\ \hline \end{array}$$

and 4+5, 5+4, and 9−4, 9−5. The terms "subtraction" and "minus," and the sign "−" are used. The pupils may be taught to think in subtracting

$$\begin{array}{r} 9 \\ -5 \\ \hline \end{array}$$

either "5 from 9 leaves 4," or "5 and 4 are 9."

Multiplication. The concepts "two times," "three times," and "four times" are developed and used in comparison of two magnitudes and two groups.

Fractions. The concepts ½, ⅓ and ¼ are developed and used. All four ideas of each fraction are developed. (See page 94.)

Geometry. The following concepts are developed and used: square, oblong, circle and cube.

Applications. All of the arithmetical ideas and facts developed this year are used in playing store, in the construction work, in games, and in the various activities of the year. Application is also made to simple one-step problems. The pupils make simple problems of their own to fit abstract statements such as 4+2=6.

Grade 2

Rote Counting. To 1000 first by 100's, then by 10's and by 1's. Counting by 2's from 1 to 11 and from 0 to 20; by threes from 1 to 13, from 2 to 14 and from 0 to 30. Counting by 4's and 5's in the same way. This work gives drill on the addition combinations and prepares for multiplication. It is both valuable and interesting but should not be overdone.

Reading and Writing Numbers.

1. In Hindu-Arabic symbols 1 to 1000.
2. In words 1 to 10, 20, 30 ... 100.
3. In Roman symbols I to XII.

Place Value. As a preparation for addition and subtraction of two place numbers and for carrying and borrowing, the pupils, through the grouping of sticks, become thoroughly familiar with the fact that a number such as 27 consists of a certain number of ones and a certain number of tens. The names and meanings of the first two places are learned. The pupils should be able to represent any two place number with sticks and also to tell what number is represented by a given combination of sticks.

Addition. 1. The combinations of the previous year are reviewed and the remaining ones are developed.

2. Columns of one place numbers not involving addition by endings.

3. Two place numbers, no carrying.

4. Zero in addition, as

$$\begin{array}{rr} 36 & 40 \\ 20 & 20 \\ \hline \end{array}$$

Subtraction. 1. The combinations of the previous year are reviewed and the remaining ones are developed.

2. Two place numbers, no borrowing.

3. Zero in subtraction, as

$$\begin{array}{rrr} 56 & 28 & 30 \\ \underline{36} & \underline{10} & \underline{10} \end{array}$$

Multiplication. 1. The multiplication ideas of the previous year are reviewed.

2. The combinations are developed and learned to 5×10. The pupils discover and make use of the principle that $2\times3=3\times2$, etc. As a preparation for the work in division and fractions each fact is drilled upon in four ways as "$2\times3=?$," "How many 3's in 6?" "3 is what part of 6?" and "½ of 6=?"

Fractions. 1. The concepts of the previous year are reviewed and used and the concept ⅙ is developed.

2. Through objective development and the work in multiplication the pupils learn ½, ⅓, and ¼ of the numbers within the tables learned.

Division. 1. The division facts are developed and memorized within the multiplication tables learned. Each fact is drilled upon in the two forms $2\overline{)6}$ and $6\div2=?$

Measures. The units of the previous year are again used and in addition the following are introduced: ounce, pound, gallon, quarter, half-dollar, dollar, dozen, peck, bushel, minute, hour, and year.

Geometry. The ideas of the previous year are used and in addition the sphere and triangle are introduced.

Applications. The arithmetic of this year is applied to the various activities of the school and to simple one-step problems. Many problems are made by the pupils (*a*) to fit given abstract statements, and (*b*) to be solved by a given process.

Grade 3

Counting. 1. 1's to 10,000.

2. 2's, 3's, 4's ... 12's, starting at 0, 1, ... within limits of addition and multiplication tables.

Reading and Writing Numbers. 1. Hindu-Arabic symbols, 1 to 10,000.

2. Roman symbols, I to XX.

Place Value. Names and meanings of first four places.

Addition. 1. Drill on combinations.

2. Addition of three and four place numbers including dollars and cents.

3. Carrying.

4. Carry combinations to higher decades as

$$\begin{array}{rrr} 4 & 14 & 24 \\ 8 & 8 & 8 \\ \hline \end{array}$$

(See page 123.)

5. Drill on column addition.

6. Addends having different number of digits as

$$\begin{array}{r} 348 \\ 27 \\ \hline \end{array}$$

7. Check by re-adding in reverse order. Make use of check habitual.

8. Use terms "addend" and "sum."

Subtraction. 1. Drill on combinations.

2. Subtraction of three and four place numbers including dollars and cents.

3. Subtraction when the subtrahend figure is greater than the corresponding minuend figure.

4. Subtrahends having fewer places than minuends, as

$$\begin{array}{r} 375 \\ 92 \\ \hline \end{array}$$

5. Check by addition. Make use of check habitual.

6. Use terms "minuend," "subtrahend" and "difference."

Multiplication. 1. Develop and drill on combinations to 10×10.

2. Two, three and four place multiplicands with one place multiplier (*a*) without carrying, and (*b*) with carrying.

3. Oral drill on $2\times3+1=?$, $3\times4+2=?$, etc.

4. Zero in multiplication, as

$$\begin{array}{rrr} 40 & 204 & 125 \\ \underline{\times\ 2} & \underline{\times\ 2} & \underline{\times\ 4} \end{array}$$

5. Check by division. Make use of check habitual.

6. Use terms "mulitplier," "multiplicand" and "product."

Division. 1. The idea of division is extended by showing the pupils that if they want to find ⅓ of a number they can do so by dividing the number by 3, etc. From this time on the two uses of division are emphasized, *i. e.*, "To find how many times one number is contained in another," (this is the pupil's fundamental idea of division) and "To find a part of a number."

2. Drill on correct responses to $\$36\div3=?$, $\$36\div\$3=?$ and $36\div3=?$

3. Develop and drill on the division facts both even and uneven. The pupils must be able to tell how many 3's in 17 and what over as well as how many 3's in 15. All of the facts a pupil must know in division are included in the following table.

How many 2's in each number from 2 to 19 and what over.
How many 3's in each number from 3 to 29 and what over.
How many 4's in each number from 4 to 39 and what over.

* * * * *

How many 9's in each number from 9 to 89 and what over.

4. Short division without carrying. Two, three and four place dividends. Form

$$3\overline{)396}\ 132$$

5. Short division with carrying (*a*) no final remainder, (*b*) final remainder. Teach pupils to express remainder as fraction.

6. Zero in division. $4\overline{)840}$, $2\overline{)406}$.

7. Check by relation Quotient×Divisor+Remainder= Dividend. Make check habitual.

8. Use terms "divisor," "dividend," quotient," and "remainder."

Fractions. 1. Review ½, ⅓, ¼ and ⅙. Develop ⅛, ⅔, ¾ and ⅝.

2. Fractional parts of numbers: (*a*) Continue objective work of second year, objective work in finding ⅛, ⅔, ¾, ¾ of numbers. (*b*) Teach finding ½, ⅓, ¼ ... of numbers by division. Memorize results within multiplication tables.

3. Addition, Subtraction and Reductions: Simple, like, unit fractions. Work out objectively ½+½=1; ⅓+⅓=⅔; ¼+¼=¾=½; ⅔−⅓=⅓; 1−⅓=⅔, etc. Memorize equivalents as ½=¾=⅝=⅘.

Measures. 1. Linear—Inch, foot and yard.

2. Money—Cent, nickel, dime, quarter, half-dollar, and dollar.

3. Dry Measure—Quart, peck, and bushel.

4. Liquid Measure—Pint, quart, and gallon.

5. Time—Seconds, minutes, hour, day, week, month, and year. Tell time. Write dates.

6. Weight—Ounce and pound.

7. Square inch. Area of rectangle by division into square inches.

8. Volume of rectangular solid by building cubic inches.

Geometry. Square, rectangle, triangle, circle, cube and sphere.

Problems. One-step problems from the text and made by the teacher. Pupils make up problems arising from their own experience. Further practice in making problems to fit given abstract statements and to be solved by

a given operation. Problems without numbers. The whole purpose of the problem work in this grade is to develop the ability to recognize addition problems, subtraction problems, etc.

Grade 4

Counting. By 2's, 3's, 4's ... 12's, starting at 1, 2 ... up to the number being used, within the addition and multiplication tables.

Reading and Writing Numbers. Numbers to 1,000,000. Roman numerals to L.

Addition. 1. Drill on combinations and addition by endings.

2. Drill on all the types of column addition of the preceding year and extend to five and six place numbers.

3. Drill on addition of columns of six or more addends.

4. Check by adding in reverse order.

5. Use terms "addend" and "sum."

Subtraction. 1. Drill on combinations.

2. Drill on examples of previous year introducing five and six place numbers.

3. Check by addition.

4. Use terms "minuend," "subtrahend," and "difference."

Multiplication. 1. Drill on facts.

2. Continue drill on types or examples introduced in third grade.

3. Develop and drill on short methods of multiplying by 10 and 100, by 20, 30, 40, etc., and by 200, 300, 400, etc.

4. Two and three place multipliers.

5. Zero in multiplier.

756	342
380	308

6. Check by division.

7. Use terms "multiplier," "multiplicand," and "product."

Division. 1. Drill on even and uneven facts, and on all the types of examples in short division of the previous year.

2. Develop and drill on short methods of dividing by 10 and 100.

3. Develop and drill on long division, two and three place divisors. All four cases. (See page 129.) Use the following form:

$$\begin{array}{r} 204 \\ 23\overline{)4692} \\ \underline{46} \\ 92 \\ \underline{92} \end{array}$$

4. Check by "Divisor×Quotient+Remainder=Dividend." Also check by estimating result.

5. Use terms "divisor," "dividend," "quotient," and "remainder."

Fractions. 1. Review fraction ideas of previous years.

2. Finding fractional parts of numbers. Review work of previous year and develop and drill on finding ⅔, ¾, ⅝, ⅜, ⅝, ⅞, etc., of numbers by division and multiplication.

3. Addition, subtraction and reductions. Review objective work of previous year. Develop principle about dividing both terms of a fraction by the same number. Drill on reducing to lowest terms. Develop and drill on rule for adding and subtracting like fractions.

4. Use terms "numerator" and "denominator."

Measures. Review and continue work of third grade. Square foot and square yard.

Problems. One and two step. Oral and written. Practical problems involving the various processes, facts and ideas taught in this grade. The developing of the ability to solve simple one and two step problems is one of the

chief things to be accomplished in this grade. The correct form of written solution is also emphasized. (See page 276.)

Grade 5

Counting. Continue work of the fourth grade, with emphasis on 6, 7, 8, and 9.

Reading and Writing Numbers. Practice reading and writing whole numbers and decimals. Roman numerals to C.

Addition and Subtraction. Continue work of previous grade.

Multiplication. 1. Continue drill on all work of previous year.

2. Develop and drill on short method of multiplying by 25 and on using known products as:

$$\begin{array}{rl} 237 & \\ 93 & \\ \hline 711 & (3\times237) \\ 2133 & (3\times711) \\ \hline 22041 & \end{array}$$

3. Teach and use check by casting out nines.

Division. 1. Continue drill on all work of previous year.

2. Develop and drill on tests of divisibility by 2, 3, 5, and 10.

3. Teach and drill on check by casting out nines.

Fractions and Mixed Numbers. 1. Develop and use idea of improper fraction and of fraction as an indicated division.

2. Continue drill on finding fractional parts of numbers.

3. Addition, subtraction and reductions. Review work of previous year. Develop and drill on (*a*) principle about multiplying both terms of a fraction by the same number, (*b*) method of finding L.C.D. by inspection, (*c*) method of adding unlike fractions, (*d*) reduction of improper

fractions to mixed numbers and vice versa, and (*e*) addition and subtraction of mixed numbers.

4. Multiplication. (*a*) Fraction by an integer; develop principles; drill on two methods. (*b*) Meaning of multiplication by a fraction. (*c*) Fraction by fraction. (*d*) Mixed numbers.

5. Division. (*a*) Fraction by an integer; develop principles; drill on two methods. (*b*) By a fraction. (*c*) Mixed numbers.

6. Cancellation.

Decimals. 1. Teach the four operations except division of decimal by decimal. Teach the following form in dividing decimal by integer:

$$
\begin{array}{r}
2.14 \\
12\overline{)25.75} \\
24 \\
\hline
17 \\
12 \\
\hline
55 \\
48 \\
\hline
7
\end{array}
$$

2. Reductions, common to decimal and decimal to common. Learn decimal equivalents of the common business fractions. (To hundredths.)

3. Check decimal point in multiplication and division by estimating answers.

4. Teach short method of multiplying and dividing decimals by 10, 100, and 1000.

Mensuration. Areas of rectangles. Develop rule.

Problems. Problems involving the ideas, facts and processes taught in this grade, particularly the work in common and decimal fractions. Emphasize the steps in solving a problem. Minimum and full forms for written solution. Check whenever possible. Form rough estimate of answer before carrying out solution. Give a great deal

of practice in planning the solution of problems. (See Chapter III, Part IV.)

Grade 6

Counting. By aliquot parts of 100 to 100. Review of counting of the lower grades.

Reading and Writing Numbers. Practice reading and writing whole numbers to a billion and decimals to four places. Roman numerals to M. Writing dates in Roman system. Enough work with decimals of more than four places to enable the pupils to learn the names of the places.

Addition, Subtraction, Multiplication, and Division. Continue drill on work of previous years. Develop and drill on tests of divisibility by 4, 6, 8, and 9.

Common Fractions. (*a*) Continue drill on work of previous year. (*b*) Drill on comparison (finding ratio) of numbers both integers and fractions. (*c*) Drill on finding whole when part is known.

Decimal Fractions. 1. Review work of fifth grade.

2. Division of a decimal by a decimal by first making the divisor a whole number. Form: Example 23.89÷3.7. Work:

$$
\begin{array}{r}
6.45 \\
37\overline{)238.9} \\
222 \\
\hline
169 \\
148 \\
\hline
210 \\
185 \\
\hline
25
\end{array}
$$

3. Develop principle "Multiplying both the dividend and divisor by the same number does not change the result."

4. Check decimal point by estimating result. In the previous example the problem is approximately 24÷4, so the result is evidently 6.45 and not .645 or 64.5.

5. Develop and drill on short methods of multiplying and dividing by aliquot parts of 100, and by 20, 30, 200, 300, etc.

Fundamentals of Percentage. 1. Concept—per cent is a number of hundredths, a fraction with the denominator 100 represented by the sign %. $17\% = 17/100 = .17$.

2. Reductions: % to common fraction, % to decimal, common fraction to % (short method and general method) and decimal to %.

3. The three problems of percentage: To find a per cent of a number, to find what per cent one number is of another, and to find the whole number when a per cent of it is known. Handle as you would the same problems in fractions. Emphasize the term base.

Examples:

1. $17\frac{1}{2}\%$ of \$23.50 = .175 × \$23.50 = \$4.11.
   $12\frac{1}{2}\%$ of 72 = 1/8 of 72 = 9.

```
  23.50
   .175
  11750
 1 6450
 2 350
 4.11250
```

2. 8 is what per cent of 12? Answer, $8/12 = 2/3 = 66\frac{2}{3}\%$.
   What per cent of 32 is 17? Answer, $17/32 = 53\frac{1}{8}\%$.

```
    .53
32)17.0
   16 0
    1 00
      96
       4
```

3. 9% of N = 63; 1% of N = 7; N = 700
   25% of N = 17; $\frac{1}{4}N$ = 17; N = 68
   17% of N = \$35; $.17N$ = \$35; N = \$35 ÷ .17 = \$205.88

4. Drill on the per cent and decimal equivalents of the common business fractions.

Applications of Percentage. Simple interest, discount, commission, per cent of increase and decrease, general applications not involving money.

Denominate Numbers. Reductions and four operations with. Two denominations only.

Mensuration. Develop and drill on methods of finding areas of parallelograms, triangle, trapezoid, and circle; and volume of rectangular solid.

Problems. Emphasize the steps in solving a problem. Minimum and full forms. Much practice in planning the solution of problems. Check whenever possible. Form rough estimate of answer before carrying out solution. (See Chapter III, Part IV.)

CHAPTER IV

TYPES OF TEACHING IN ARITHMETIC

No book can solve for the teacher all of the problems that will arise in the teaching of arithmetic. The best it can hope to do is to help the teacher solve her own problems. Good teaching can never come from the mechanical and slavish following of any set method. The teacher must always be an intelligent, thinking being, who, trained in the technique of her profession, and guided by a working, functioning knowledge of general principles, thoughtfully and intelligently, meets each difficulty as it arises.

Definition of Method. Most teachers and many books on methods have too narrow a conception of the meaning of method. A method is simply a means to an end. With the end in view, the method is the tool used to realize this end. *There is no one best method of teaching any part of the subject of arithmetic.* The value of methods is purely relative and, as previously stated, can be measured only in terms of the particular ends to be accomplished. The method that is best for one class or one teacher is not necessarily best for another class or teacher. For any given teacher, that method is best which, for that particular teacher, with her particular class and with all the special conditions surrounding that class, will most nearly accomplish the end that should be accomplished by the teaching of that particular piece of subject matter.

Purpose of This Book. This book, therefore, will make no attempt to consider in detail the best method of teaching this, that, and the other topic, throughout the whole subject. Its purpose is rather to describe the different kinds, or types, of work to be done in the teaching of arithmetic,

to give an understanding of the underlying psychology, and to aid the teacher in gaining a mastery of the technique of each of these types.

Several excellent books have been written on the types of teaching that arise in teaching in general. It is the purpose of this book to apply the general discussion of such books to the particular subject of arithmetic.

Types of Teaching. In general, the teacher of arithmetic has three things to do. (1) Present new knowledge. (2) Fix in mind and make habitual and automatic knowledge already presented, and (3) develop in the pupils the ability to apply this knowledge to concrete, life situations. It must not be supposed that these types of teaching necessarily occur separately, and that they are mutually exclusive—that in one recitation the teacher is always concerned with one type only and in another recitation with another type. All of these types may occur in a single recitation, but it will simplify the discussion and also make it easier for the teacher to understand each type if we first consider them separately.

Presentation of New Knowledge

Kinds of Knowledge to be Presented. The new material to be presented in arithmetic is of several kinds. It may be a new general *idea* or *concept* such as, the notion of what is meant by "7," "½," "square," or "addition." It may be a new *fact* such as, "2+3=5," "3×7=21," or "C=πd." It may be a *principle,* that is, a truth of a more fundamental and basic kind such as, "Multiplying both the numerator and denominator of a fraction by the same number does not change its value," or "The product of two or more factors is the same no matter in what order the multiplications are performed." It may be a *process* such as column addition, long division or square root. Finally it may be a *rule,* such as "To divide by a fraction,

invert the divisor and multiply," or "To multiply decimals multiply as with whole numbers and point off in the product as many decimal places as in the multiplier and multiplicand together."

Fixing and Mechanizing

In arithmetic every recitation does not add to the pupils' knowledge. The purpose of many recitations is not to teach something new but to fix in mind and mechanize knowledge that the pupils already have in their possession. This type of work is usually called *drill work,* or *drill.*

Developing the Ability to Apply

It is not enough that the pupils know how to add, subtract, multiply and divide; they must also know when to use these processes and facts. The purpose of much of the work in arithmetic is to develop in the pupils the ability to apply their knowledge of arithmetic to concrete situations.

Problems and Examples. The concrete situation or its description is brought into the school room as a *problem.* The term problem will be used in this book in the above sense, namely, to mean a concrete life situation having a numerical side. For the abstract drill situation such as, $47\overline{)3978}$ or $\begin{array}{r} 389 \\ 756 \\ \hline 832 \end{array}$ the term *example* will be used.

PART II

THE PRESENTATION OF NEW MATERIAL

CHAPTER I

METHODS OF PRESENTING NEW MATERIAL

Possible Methods. In presenting new material to pupils four methods are possible. The teacher may (*a*) give arbitrarily without justification or explanation, (*b*) give and then explain—that is, by a line of logical reasoning attempt to convince the pupils of the correctness of the fact, rule, or process already presented, (*c*) give and then have the pupils verify the truth of the fact or the correctness of the result obtained by the rule or process, or (*d*) lead the pupils to discover the new for themselves.

These four methods may be illustrated by the presentation of the rule for multiplying an integer by 10, namely, "To multiply an integer by 10, annex a cipher to the right." According to the first method, the teacher tells the pupils, arbitrarily, how to multiply an integer by 10 and proceeds at once to drill them on the process.

In using the second method the teacher first presents the rule arbitrarily, as before, but then tries to convince the pupils of the correctness of the rule by explaining it in terms of place value, namely, "In the system of writing numbers which we use, the value of each place, as we go from right to left, is ten times that of the preceding place, thus, the first place is units, the next tens, the next hundreds, etc. So annexing a cipher at the right pushes each digit one place to the left, that is into a place, the value

of which is ten times as great as the value of the place the digit originally occupied. Therefore the value represented by each digit and the whole number is multiplied by 10." Such a method of presentation attempts not only to convince the pupil of the correctness of the rule but also tries to make clear *why* the rule is correct.

The third method aims only at convincing the pupils that the rule is correct by having them verify the answer obtained by using the rule; it does not attempt to show *why* the rule gives the correct result. The teacher presents the rule arbitrarily as before and then requires the pupils to verify the result, in this case probably by addition. Using the rule the pupils get $10\times23=230$, then by adding ten 23's verify the correctness of the result and of the rule.

The fourth method is radically different from the other three. In using it the rule comes last instead of first and is stated by the pupils instead of by the book or by the teacher. From their knowledge of the multiplication facts, the pupils already know that $10\times7=70$, $10\times3=30$, $10\times9=90$, etc. If their attention is called to these facts they can easily be led to compare and generalize for themselves, concluding that if they want to multiply any number by 10, all they need to do is to annex a cipher. Then they can verify the results obtained by using their rule as in the preceding method. This is the method of *inductive* development or the method by which the pupils are led to discover general truths by examining and comparing special cases. Or, the new knowledge may be obtained by using general knowledge that the pupils already have in their possession. This is the method of *deductive* development. Inductive and deductive development are alike in that they both start with a problem or difficulty; they differ only in the means taken to meet this difficulty. In both, the new rule, principle, process or fact comes last as a result of the thinking and not first.

These methods of discovery will be treated in detail in the succeeding chapters.

The "Give" Method. This method has two decided advantages—it is the quickest and easiest way of presenting new material. The chief objections to the use of this method are, (1) telling does not insure understanding and (2) it tends to form in the pupils the habit of depending on others for their knowledge rather than the habit of thinking things out for themselves. There can be no question as to which habit is the more useful in life and consequently which the schools should encourage. Probably the greatest defects of much teaching in the past have arisen from regarding teaching as a giving process and learning as a receiving process. Modern education is no longer a process in which the teacher pours information into the pupils but rather one in which the pupils learn and grow by discovery and by doing things for themselves.

The "give" method is utterly opposed to the spirit of modern education and should seldom be used. It is the resort of the lazy or poorly trained teacher who knows no other way.

The "Give and Explain" Method. This method is the result of the old scientific and disciplinary teaching of arithmetic. There are several objections to its use. Pupils in the elementary school, particularly in the lower grades, are very much interested in knowing *what* is true but not usually in *why* it is true. Further, a logical explanation is often beyond their comprehension. Such explanations should be given only when demanded by the pupils. After the pupils are convinced that annexing a cipher multiplies by ten, if they raise any question as to why it does so, the teacher should explain, otherwise no explanation should be attempted. This method also makes the pupils passive participators in the learning process as

the explanation must usually come from the book or teacher and at best all the pupils do, is to follow. Usually they do not do even this, for if they are already convinced of the correctness of the rule, they do not see the need of any explanation and therefore are not interested. If they are not convinced, the explanation will not usually convince them. The too frequent use of this method is the mistake often made by the young teacher just out of college who attempts to use in the elementary school methods suitable for the university.

The "Give and Verify" Method. The third method is open to the objection that after they have been given the fact, rule, or process by the book or teacher, most pupils are perfectly willing to accept it, and do not see the need for any verification. If, however, the new fact is discovered by the pupils, the need of verification immediately arises and the pupils are very much interested in testing the correctness of their own conclusion.

The Method of Discovery. This method has many advantages over the other three. In the first place, it secures thinking on the part of the pupils because it presents the proper situation to cause thinking—that is, a problem situation. None of the other methods do this as they all *start* by giving the information. In this method, the pupils are confronted with a problem. In the illustration used before, the problem is to discover an easy way to multiply by ten. In order to solve this problem, the pupils must review what they already know about multiplying by ten, compare, draw their conclusion and verify. This method has the further advantage that it encourages the pupils to use what they already know to meet new situations, it trains them to acquire knowledge for themselves, to verify their first conclusions, and not to depend entirely on their teacher and text-book. Finally it is more interesting, simply because of the greater activity it demands

of the pupils and because of the natural pleasure of discovery and of doing things.

Conclusions. Whenever possible, pupils should be led to discover new knowledge for themselves and should then be required to verify their discovery. Then if any explanation of why things are as they are is demanded, it should be given by the teacher for the benefit of those pupils who see the need for and are interested in it. If for any reason such as lack of time or too great difficulty, the pupils can not discover new information and must get it from the teacher or book, they should be asked to verify so they will not form the pernicious habit of believing everything they read or hear.

CHAPTER II

THE INDUCTIVE DEVELOPMENT LESSON

The process by which the mind discovers new general truths by studying and comparing individual objects or cases is called *induction*. It is the method usually used in discovering new knowledge in arithmetic. The discovery of the rule for multiplying by 10 from the multiplication facts, $10\times3=30$, $10\times9=90$, etc., given in the preceding chapter, is an example. New ideas, facts, principles and processes may be discovered in the same way. For example, by counting, the pupils find that 4 apples and 5 apples are 9 apples and that 4 boys and 5 boys are 9 boys and so on, and come to the conclusion that 4 and 5 are always 9. Or, by measuring the circumference and diameter of a circular object and comparing, the pupils find that the circumference is a little more than three times the diameter. If it is then suggested to them that they measure other circles of different sizes, they will find that the same relation still holds and finally come to the general conclusion that no matter what the size of the circle the circumference is always a little more than three times the diameter.

It must be remembered that induction is a means of discovering apparent truth and not a means of proof or explanation. The fact that a certain thing is true ninety-nine times is no proof that it will be true the hundredth time. To get a rigorous, logical proof or explanation, another type of thinking must be employed. But, as has already been pointed out, the pupils in the lower grades do not feel the necessity for, nor are they interested in logical proofs and explanations. They are in the stage of mental exploration and discovery.

The Steps in the Inductive Development Lesson. In arriving at a general conclusion by conscious induction, the mind goes through certain steps or phases of thinking which are roughly as follows: (1) The realization and definition of the problem, (2) The selection of individual cases coming under the general problem, (3) The study and comparison of the individual cases, picking out the essential elements or those bearing on the problem to be solved, and eliminating the non-essential, (4) The forming of an hypothesis or tentative generalization and the verification of this hypothesis, (5) The final acceptance and formulation of the generalization, (6) The application of the generalization to be sure everything is thoroughly understood. The phases of the inductive lesson corresponding to these phases of inductive thinking will be spoken of hereafter, for convenience, as, (1) The Problem, (2) Selection of Cases, (3) Comparison and Abstraction, (4) Hypothesis and Verification, (5) Final Generalization, (6) Application.

The Problem

The purpose of this phase of the recitation is twofold, (*a*) to show the pupils that their present store of knowledge is insufficient to enable them to meet successfully some new situation or situations and to convince them of the importance and necessity of being able to master such situations, and (*b*) to make clear to them just what new knowledge they will need in order to do so. These might be called *Motivation* and *Definition* of the Problem.

Motivation. To motivate the presentation of new knowledge means to convince the pupils that they need new knowledge in order to accomplish some end in which they are interested, to show them that their further advancement in the subject depends upon acquiring the new information.

The teacher has the power to give the pupils the information whether they want it or not but she hasn't the power to make them take it. One is reminded in this connection of the old saying, "You can lead a horse to water but you can't make him drink." Pupils will be much more interested in acquiring new information and it will mean much more to them if they feel a need for the new knowledge. A little time spent in establishing a need or providing a motive will later be saved many times. The psychological time to present any new topic is when the pupils feel a need for it and, with a properly organized course of study and text, a skilful teacher can usually make the need arise naturally out of the work the pupils are doing. When this can be done better results are obtained in less time.

The ultimate motive for all of the work in arithmetic must be sought in its practical applications which are brought into the school-room in the form of interesting and real, life problems. If the pupils can be interested in some practical application of arithmetic, in some problem, and find that they can not solve this problem because they do not know how to multiply decimals, the need of gaining this new knowledge is brought strongly home to them.

As an illustration, take the following development of the rule for multiplying a fraction by a whole number, which came under the observation of the author recently. A group of girls in their work in cooking had encountered the following situation. A recipe called for ¾ cup of flour and made enough for three people. Their problem was to enlarge the recipe so as to make enough for six people. The cooking instructor had referred the difficulty to their regular teacher. The girls realized that their stock of arithmetical knowledge was insufficient and were very anxious to add to it so they could meet this situation.

Although the problem motive is the best and most universal, there are others that can also be used effectively. In the lower grades, particularly, the game motive is often very effective. The pupils can be interested in some game, the playing of which involves something that they do not know, and in order to play the game they are interested in acquiring the new knowledge.

A short review can often be used to show pupils that they need to know something new. The teacher can ask the pupils to make a summary of the arithmetical knowledge that they possess along some particular line and have them examine it to see if there is anything lacking. As an illustration, suppose a class is ready for the new process of dividing a decimal by a decimal. The teacher might motivate the new work by having the pupils summarize what they already know about division of decimals, namely, (*a*) how to divide a decimal by an integer, and (*b*) how to divide a decimal by powers of ten. If the teacher then asks the pupils what they do not yet know in division of decimals, the need of the new knowledge, *i. e.*, how to divide a decimal by a decimal, will be clearly seen.

It has just been seen how work can be motivated by review or by looking back. It can also often be motivated by looking ahead to what is coming. As an illustration, suppose pupils have just started the work in addition of fractions with unlike denominations. The first cases met are simple ones such as, $\frac{1}{2}+\frac{1}{4}$ and $\frac{1}{2}+\frac{1}{3}$. In these simple cases the pupils depend on memory to tell them what the common denominator will be. Thus they know in $\frac{1}{2}+\frac{1}{4}$ that the common denominator is 4. But later they are going to meet cases like $\frac{3}{8}+\frac{7}{12}$, in which they will not know the common denominators. By looking ahead and considering such cases with the pupils the need for a method of finding the common denominator becomes evident.

The pupils can also be shown the need of an easier

way of doing something. For example, they may know how to multiply $22\frac{1}{2} \times 3\frac{1}{4}$ as common fractions and they can be shown that the process is a laborious one and told that it is often easier to write mixed number as decimals and then multiply. This could be made the motive for finding out how to multiply decimals.

Finally, new work may sometimes be motivated by making an appeal to the pupils' pride. The desire to be able to work with larger numbers or to make larger scores in the games is often sufficient motive. Suppose third grade pupils have been playing the bean bag game* and instead of taking the number hit as their score have been multiplying the number hit by 3. The desire to make larger scores in this game would then be sufficient motive for learning the multiplication table of 4's. In a rural school or in a room in which two grades or two sections of the same grade are seated, pupils in the lower grade or section often want to learn something new simply because they have seen the older pupils using it. The author once knew a third grade pupil who demanded that his teacher teach him long division. He had become interested in it through watching the fourth grade pupils in the same room.

Definition. In the situation described above the girls felt the need of increasing their arithmetical knowledge in order to solve the problems that arose in their cooking class. Thus the motive was provided for the arithmetic teacher. She still had to make sure, however, that the girls knew exactly what the difficulty was, namely, that they could not solve the problem because they did not know how to multiply a fraction by a whole number. Teachers often make the mistake of trying to lead the pupils to discover something when the pupils have only a vague idea of what it is they are to discover. Before

*See page 180.

trying to find a method of overcoming the difficulty, the teacher must be sure that the pupils know exactly the nature of the difficulty to be overcome. The problem must always be definitely stated, preferably by the pupils.

Selection of Cases

Simplicity. Having defined the difficulty and aroused the pupils' interest in overcoming it, the next step is the selection of individual cases from which the pupils are to discover their solution of this difficulty. Thus in developing the rule for multiplying a fraction by an integer the cases $2\times\frac{1}{5}$ and $3\times\frac{3}{10}$ might be used. The individual cases should be carefully chosen. In the first place, they should be just as simple as possible. In developing the rule mentioned above, to use cases such as $8\times\frac{7}{5}$ would simply serve to increase the difficulty and distract the pupils' attention. In presenting the rule for multiplying decimals, $.2\times.3$ will bring out the essential points better than 3.75×37.9, for it is easy to see at a glance that 6 is the product of 2 and 3, but it is not so easy to see that 142125 is the product of 375 and 379.

Variety. In the second place, the cases used must have sufficient variety to bring out clearly the essential points and to make the conclusion correct and general. If in developing the rule for multiplying a fraction by a whole number, the cases $2\times\frac{1}{3}$ and $3\times\frac{1}{3}$ are used, the pupils might draw the incorrect conclusion that the numerator of the answer is always the same as the multiplier.

Typical. Further the cases used must be such that the essentials stand out prominently. If, in developing the rule for multiplying a fraction by a fraction, the case $\frac{2}{3}\times\frac{6}{7}$ were used, the pupils might think it through and get the result $\frac{4}{7}$ which is certainly correct, but does not bring out the essential relations sought, namely, that in multiplying fractions the numerator of the result is the

product of the numerators, and the denominator of the result the product of the denominators.

Number of Cases. No arbitrary statement can be made as to the number of individual cases that should be used in this step. *Enough must be used to bring out the essentials sought; more than that is a waste of time.* Sometimes a single case is sufficient but if a tentative generalization is drawn from a single case, the verification becomes doubly important and several cases should be used in that step.

Comparison and Abstraction

In this step the pupils must be led by the teacher to see the essentials in each of the individual cases presented and, by comparison, to pick out the relations that are common to all. Teachers sometimes perform this step for the pupils, thus largely destroying the value of the whole lesson. The pupils can abstract the essentials for themselves *if they have a skillful guide.* If, in developing the rule for multiplying a fraction by an integer, having used the two cases, $2\times\frac{4}{9}=\frac{4}{9}+\frac{4}{9}=\frac{8}{9}$ and $3\times\frac{3}{10}=\frac{3}{10}+\frac{3}{10}+\frac{3}{10}=\frac{9}{10}$, the teacher should ask a general question such as "What do you see that is alike in both these cases?" the pupils would not see anything. But by proposing the definite problems, "Let us see if we can find a short way of getting the answer without having to add," "Who sees how we could get the numerator of the answer in each problem?" "How could we get the denominator?" and "Who can tell me how to multiply a fraction?" the pupils will have no difficulty in picking out the essential relations for themselves and forming their own rule.

The comparison and abstraction may often be aided by the elimination of non-essentials and by the arrangement of the work. Thus instead of comparing the two cases mentioned above as they were first worked out, namely,

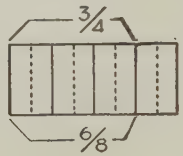
$2\times\frac{1}{9}=\frac{1}{9}+\frac{1}{9}=\frac{8}{9}$ and $3\times\frac{3}{10}=\frac{3}{10}+\frac{3}{10}+\frac{3}{10}=\frac{9}{10}$, the teacher might eliminate the intermediate step by which the answers were obtained and rewrite the problems and answers only ($2\times\frac{1}{9}=\frac{8}{9}$ and $3\times\frac{3}{10}=\frac{9}{10}$) placing one under the other to facilitate the comparison.

Hypothesis and Verification

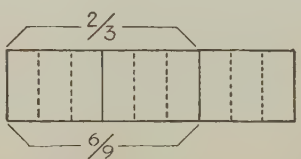
The first generalization made by the pupils should not be final, but tentative, rather in the nature of an hypothesis than a fixed, final conclusion. It seems to be human nature to generalize from too few data and the schools should do all in their power to counteract this tendency instead of encouraging it as they sometimes do. It is impossible in the schools to have pupils examine hundreds of different cases before drawing any conclusion, but the teacher can prevent their forming the habit of jumping to a fixed, final, general conclusion as soon as they see that a certain relationship exists in one or two cases. When they see, for example, that if they take the fraction $\frac{2}{3}$ and multiply both terms by 2, the resulting fraction $\frac{4}{6}$ has the same value as the original fraction $\frac{2}{3}$, they should not be led or permitted to come to a final conclusion as none is justified as yet. They should simply be led to think "Why, that is odd. I wonder if that is always true?" and should then proceed to find out by examining other cases. If they find that the same relationship holds in several other cases, they are then justified in forming the conclusion that "Multiplying both the numerator and denominator of a fraction by the same number does not change the value of the fraction." If any doubt remains the teacher and the book can assure the pupils that the relation that they have discovered is universally true. The habit of inductive thinking, to be formed in school and used in life should not be (*a*) Notice relationship in one or two instances, (*b*) Conclude that this relationship always exists, but

(*a*) Notice relationship in one or two instances, (*b*) Wonder if the same relationship always holds, (*c*) Proceed to verify the supposed discovery and (*d*) Draw conclusion. In life this conclusion should not even then be final but subject to change in case instances are met that do not conform.

Methods of Verifying. Two methods of verification are useful in arithmetic, (*a*) by examining more cases, and (*b*) by checking in some way the results obtained by using the tentative conclusion. Suppose pupils have taken the fraction ¾ and multiplied both terms by 2 and, by representing both graphically, found that the resulting fraction ⅝ is the same as the original fraction ¾ and have formed



the hypothesis that, "Multiplying both the numerator and denominator by the same number does not change the value of the fraction." Then they could verify this hypothesis by taking the fraction ⅔, multiplying both terms by 3, getting ⅙, and representing both fractions graphically.



Other similar cases might also be taken.

Or, having, from the facts $10\times7=70$, $10\times5=50$, $10\times3=30$, drawn the tentative conclusion "To multiply a number by 10 annex a cipher" the pupils could verify

this conclusion by applying it to 10×17 and checking the result obtained by addition.

$$
\begin{array}{r}
17 \\
17 \\
17 \\
17 \\
17 \\
17 \\
17 \\
17 \\
17 \\
17 \\
\hline
170
\end{array}
$$

Final Generalization

Made by Pupils. In this step, the general fact, principle, rule or process discovered from the individual cases studied and compared in the previous steps, is formulated and stated in clear, concise form. The generalization should be made by the pupils, not by the teacher. The pupils' first statements will be crude and incomplete, but by comparing and criticizing the statements of one another, they can, with some help from the teacher, formulate their conclusion in simple, brief form.

The author recently observed a recitation in which the class discovered the rule for multiplying by ten. After taking several cases, the teacher asked, "Who can tell us the shortest and easiest way to multiply by 10?" The first statement of the rule given by a pupil was, "The easiest and shortest way to multiply by 10 is to add a nought." Another statement was, "To multiply any number by 10 you just put a nought after it." The pupils then decided that the "you" was unnecessary and finally stated the rule in the form, "To multiply any number by 10 add a nought."

In Words and Symbols. In many cases the generalization should be made not only in words but also in mathematical symbols. Thus, by counting, the pupils find that three boys and four boys make seven boys, three apples and four apples make seven apples, and draw the conclusion that three and four are seven, expressing the conclusion in words and then in symbols $3+4=7$. Similarly, by measuring, the pupils discover that the circumference of a circle is a little more than three times the diameter and state the conclusion in words and symbols, as $C=\pi d$.

Application

To Problems—Purpose. The step of application has a double purpose. Having acquired new information, the test of whether the pupils really understand and appreciate its full significance is their ability to use it. In arithmetic it is not sufficient to possess certain knowledge but one must know how to use this knowledge as a tool in solving concrete, arithmetical situations. Thus, application is the final step in acquiring new information. It is not enough for the pupils to know that $3+4=7$ or that $C=\pi d$; they must also know when to use these facts. The new fact or other information must be connected *from the beginning* with the kind of concrete situations in which it is used. Therefore it is not sufficient to use abstract drill examples in the step in application. In order that the pupils may understand the importance and meaning of the new information and know in what kinds of situations to use it, application must be made to concrete, applied problems.

To Examples—Purpose. Besides being the final step in the acquisition of new knowledge, the application is also the first step in making this information a permanent possession of the mind. The pupils, having discovered how to multiply decimals would forget before the next

day if they did not at once proceed to use the rule, at least a few times. The application to applied problems helps fix the information in mind but usually this is not sufficient and, therefore, abstract drill examples must also be used for this purpose.

In the step of application, the pupil is applying the general knowledge, that he has just discovered, to particular cases—that is, the thinking involved is deductive. In spite of this, however, this step is an integral and necessary part of the inductive development lesson and should not be separated from the other steps. Since the last step of the inductive lesson is really deductive, such a lesson is often called inductive-deductive.

Supervision. The first applications made by the pupils must be made in class and closely supervised by the teacher so that she can correct any mistaken ideas the pupils may have before they become fixed in mind.

To assign seat or home work for the pupils at the close of the step of verification and before the pupils have had a chance to use the newly acquired knowledge under the supervision of the teacher is to invite trouble. The teacher should be sure that every individual pupil can not only state the generalization but also *use* it before it is safe to ask the class to use it for themselves.

The Steps Are Not Separate

It should not be thought that the steps discussed above are absolutely separate and distinct. "Their arrangement in a definite order is more or less formal, useful for the purpose of description. They overlap and inter-penetrate one another in the actual thinking process. ... While the idea of inductive method as a series of steps, each one complete in itself before the next is entered upon, breaks down; it still remains true that there are certain characteristic phases or movements of thought, in the inductive

process each one of which is necessary in some degree of its fulfillment to the next."*

Time Required

Teachers often feel that they have not time for inductive teaching but must resort to telling. This attitude usually arises from the fact that the subject matter is more important in the teacher's mind than the effect the method of presentation will have on the pupils. This is largely the result of the old scientific attitude. Inductive work is so important that the teacher must find time for it even if she finds it necessary to sacrifice such topics as cube root, and Longitude and Time. As a matter of fact, the inductive presentation of new material does not take nearly as much time as many teachers think. Often the first five steps can be covered in a few minutes and the rest of the recitation spent on application. Sometimes, however, such a recitation will require more time. The problem may arise and be defined one day and the other steps taken the next. Often this is the better way. In the work of today, the pupils encounter a new difficulty and become interested in overcoming it, but if they have not sufficient time they can postpone the solution of the problem until the next day. Sometimes it may even be necessary to postpone the final step of application until a third recitation; whenever possible, however, at least a little time should be given to the application in the same recitation that the discovery of the new knowledge is first made.

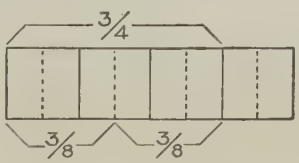
Thus the time taken for the inductive presentation of new knowledge varies from just a few minutes to two or three recitations, but in any case the time is well spent as the pupils get at least as much real benefit from mastering the technique of discovering new knowledge for themselves as they do from the particular fact or rule discovered.

*Miller, *The Psychology of Thinking,* pp. 249-50.

CHAPTER III

THE INDUCTIVE DEVELOPMENT LESSON —OBJECTIVE WORK

Much of the inductive work of the grades is also objective, that is, the individual cases from which the pupils get the new general knowledge are concrete, objective situations. Thus the pupils get the fact $3\times4=12$ by taking 3 groups of 4 objects each, using various kinds of objects, until they discover that 3 fours always make 12; they find that the circumference of a circle is always a little more than 3 times the diameter by measuring circular objects; and they learn that multiplying the denominator of a fraction divides the value of the fraction, by taking some fraction as ¾, multiplying the denominator by 2 and picturing both the original and resulting fraction graphically or objectively as:



Purpose of Objective Work

To Develop Ideas. The use of objective situations is particularly important in the acquisition of new ideas and facts. The teacher can not give the pupils the idea "one-half," for example, by telling them what it means. The only way the pupils can really get the idea is through their own experience by taking objects and groups of objects and dividing them into two equal parts and then being given the name of these equal parts.

To Develop Facts. It is also impossible to convey a new kind of fact abstractly so that it will really mean anything to the pupils. The teacher can tell the pupils that 2 and 3 make 5 but unless they actually work out this or similar facts objectively, it will mean nothing to them. The story is told of the little boy who came home from school one afternoon and was asked by his mother what he had learned that day. He replied that he had learned a poem about the Tootums Family. When his mother asked him to repeat it he did so as follows:

Tootums one is two,
Tootums two is four,
Tootums three is six,
Tootums four is eight.

He had been learning multiplication facts but had no idea of their meaning and regarded them simply as a poem. This story may or may not be true, but it illustrates a situation that often occurs—pupils memorizing a set of abstract facts that have absolutely no meaning to them because they have never been met objectively.

It is essential that ideas and facts that are entirely new be first met in objective situations in order that the abstract words and symbols stand for real ideas and call forth the proper imagery in the pupils' minds. This is the purpose of all objective work in arithmetic.

When to Use Objects

The tendency in the past, where objects have been used, has been to use them extensively in the lower grades and not at all in the intermediate and upper, on the supposition that the older pupils do not need such objective treatment but can grasp the idea or fact abstractly. Such an assumption is false. The mature mind is no better able to get an entirely new idea abstractly than is the immature mind. Objective presentation is just as essential in the university

as in the primary grades. The only reason objects are used more extensively in the primary grades is that the pupils at that stage are acquiring more new ideas and facts, than at any other time.

When to Stop Using Objects

It is just as essential to know when to stop using objective situations as it is to know when to use them. The pupils must eventually know that $2+3=5$ and that $3\times5=15$ without having to use objects or even think in terms of objects. It is not necessary to present all the addition, subtraction, multiplication, fraction and division facts concretely; indeed it would be a waste of time and would defeat the desired ends, as the pupils might form the habit of thinking objectively and be unable to handle the abstract number facts. It is, however, essential that the first addition, subtraction, multiplication, fraction and division facts be met in concrete setting in order that these types of facts may have meaning to the pupils. The objective situations should be used only until the pupils have grasped the meaning of these facts. It is just as bad to keep the pupils in the objective stage of development too long as it is to neglect it entirely.

Characteristics of Good Objective Material

Variety. The material used should be carefully chosen. There are several requisites for good material. In the first place, it must be varied. If the pupils work out all the number facts with sticks, there is danger that they will associate them with sticks only and be unable to apply them to situations involving other objects. The author recently observed a recitation in which the teacher was trying to develop the idea of a square. The pupils were shown several squares made of cardboard and of different sizes and colors. They found that the figures had four

sides and four corners and by measuring found that the four sides of each figure were equal. The teacher then gave the name square and asked the pupils to find all the square objects in the room. They readily picked out a square blotter on the teacher's desk and a square picture mounted on a square piece of cardboard, but failed to recognize the square window panes, the squares on a large calendar on the wall or the squares in the iron ventilating register. They had the idea square associated with cardboard and could not apply it to other things.

Further, in seeking variety, the teacher should not confine herself to three dimensional objects. Pictures, repetition of acts, dramatization, games, measuring, marks, number pictures, diagrams and models are objective situations as well as sticks, pencils, cubes, books and desks. The pupils can get the idea of 7 from clapping their hands seven times or measuring a line that is seven feet long just as well as from counting seven three dimensional objects. Indeed, they will get a much better and broader idea if they do all three things.

Reality. Besides being as varied as possible, the objective material used must be real to the pupils. It is much better, particularly in the beginning, to use books, pencils, desks, the pupils themselves, hair-ribbons, buttons on dresses, toy money and standard units of measure than sticks, cubes, beans and lentils. The real objects are more interesting and possess the additional advantage that the pupils acquire the ideas and facts from the kind of material to which they are going to apply them and not from highly artificial and abstract material.

Unity. Although a variety of material must be used in presenting any new idea or fact, it is better, when possible, to have all the material used in a given recitation connected and unified in some way. For example, in a given recitation toy money, scales and weights, yard sticks, boxes

of crackers, and other merchandise might all be used and still all be grouped around the central situation of Playing Store. The material might also be unified by being used in the dramatization of some story, problem or picture; in constructing something in Industrial Arts; or in playing some game. Such unification of material is desirable, as it makes the work both more interesting and more real.

The Kinds of Objective Situations That Should Be Used

The following kinds of objective material have been suggested in this chapter:

1. Three dimensional objects that are real and interesting to the pupils, such as toy money, the merchandise in the play-store, the standard units of measure, the pupils themselves, bean bags and other objects used in their games, and the common objects of the school-room such as books, pencils, erasers, desks, etc.

2. Pictures in the reading primers and other books, pictures on the wall and picture cards or charts illustrating the primary number facts. These can be bought or made by cutting out pictures and pasting them on cardboard. For example, to illustrate the number fact $3+5=8$, the picture of eight apples could be cut from a fruit catalog and pasted on a card in two groups of three and five, respectively.

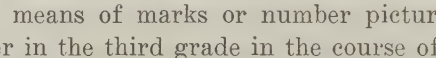
3. Repetition of simple acts such as clapping hands, taking steps and knocking on the door.

In addition there are several types of material that might be called semi-objective:

1. Conventional, artificial objects, such as cubes, sticks, beans, etc. These possess one decided advantage, namely, convenience. It is possible to have an unlimited supply of them always on hand; they are easy to handle and economical of time. In the beginning, these objects can

be made more real and interesting by imagining that they represent real things. Thus in dramatizing the problem—"If Johnny had three apples in his lunch basket this morning and ate one at recess, how many has he left?"—the pupils could use sticks and imagine they were apples. This make believe appeals to the instinctive tendency of the child to pretend and imagine, and often makes the situation just as real and interesting as if the real objects (in this case apples) were used. Later, to save time, these objects can be used simply as objects without thinking that they represent anything in particular.

2. Number Pictures and Marks. Even the conventional objects are not always at hand when needed but the teacher and pupils can always picture the number situation, objectively, by means of marks or number pictures. Thus if the teacher in the third grade in the course of a recitation has reason to believe that one of the pupils does not know what $3\times4=12$ means, she would need to objectify the situation to make its meaning clear. She might not have any objects convenient but could have the pupil represent it by marks on the blackboard as,



3. Diagrams and Models such as the well known fraction diagrams as,

the diagrams and models used in teaching mensuration, square and cube root blocks, etc.

Conventional objects used simply as abstract objects, number pictures, diagrams, and models all have the advantage of being always available and very economical of time but they have the disadvantage of being less objective

—more abstract than the other types. For this reason their use should be avoided at first but as the child develops they may be used to save time and they further serve as a transition from the concrete stage to the abstract symbolic stage.

The objective work can be made meaningful and real if it is unified around some life situation and if the various types of objects used grow naturally out of that situation. The three types of situations most useful for this purpose are:

1. Dramatization of stories, pictures and problems; and the playing of store, postman, bank, stock exchange, etc. These are not only objective situations themselves but also unify and make real the large variety of objects used in the dramatization of the complex situation.

2. Games also present a real, concrete situation and serve to unify the various objects used. For discussion of games, see Chapter VI, Part III.

3. Measuring and Construction Work. Much of the arithmetic of the primary grades should be correlated with and grow out of the work in Industrial Arts or the construction work. This work constantly presents new number ideas and facts in a concrete way and unifies and renders meaningful the various types of objective material used.

CHAPTER IV

THE DEDUCTIVE LESSON IN ARITHMETIC

Definition. We have already considered the type of thinking through which the pupils discover general truths by the observation and examination of particular cases. As soon as they have acquired even a limited stock of general knowledge, another kind of thinking becomes possible, namely, the type of thinking in which they overcome a new difficulty by the use of general truths that they have already accepted as valid. Such thinking is called *deductive thinking,* or *deduction.* It may arise in arithmetic in three different ways according to the nature of the difficulty to be overcome, but it is always the same in that the difficulty is met not by the study of individual cases as in induction but by the direct application of general principles.

Three Types of Deduction in Arithmetic

The Solution of Problems. The difficulty, or problem, may be some concrete, arithmetical situation which is met by applying general knowledge. Thus when the pupils meet the situation of determining how much three articles will cost at five cents each, they solve it by using their knowledge that 3×5 is always 15 and do not have to resort to counting. The solution of every applied problem in arithmetic involves this type of deductive thinking and it will be considered in Part IV in connection with the discussion of problem solving.

Deductive Discovery. Again the problem may be to discover some new truth. General truths are not always discovered by examining individual cases but may also

be discovered by putting together and rearranging known truths in new ways. This type of deductive thinking is often used in geometry. For example, the student, by making use of old knowledge (axioms, definitions, previous propositions, etc.), arrives at the new knowledge that the angles opposite the two equal sides of an isosceles triangle are equal, without having to examine and measure the angles of a large number of such triangles.

Deductive Explanation. Finally the problem may be to verify or explain some general truth previously discovered and accepted. Some doubt may have arisen as to the general validity of the truth or the pupils may have raised the question as to why it is true. In order to prove that a truth is universal, or to explain why it is true, it is necessary to show that it is the result of other truths already accepted as valid. The thinking involved is thus deductive.

Deductive Discovery—The Deductive Development Lesson

As soon as the pupils have a fund of general knowledge, it is possible for them to arrive at new truths by combining and rearranging old without having to resort to the observation and study of individual cases. The type of recitation in which the pupils are thus led to formulate new truths from old is usually called the *Deductive Development Lesson* although, as will be seen later, it usually involves induction as well as deduction.

Steps in the Deductive Development Lesson. The steps or phases of the deductive development lesson are:

(*a*) The problem—to discover some rule, principle, process or fact.

(*b*) The recall and combination, under the guidance of the teacher, of the general truth or truths of which the new truth is the consequence.

(*c*) The formation of a tentative conclusion or hypothesis, and the verification of the supposed truth through induction, that is, by the examination of particular cases to see if the supposed truth holds.

(*d*) The final generalization.

(*e*) The application of the truth.

The inductive and deductive types of discovery both begin with a problem, form an hypothesis, and verify it, formulate a general conclusion and then apply it. These steps are identical in the two types of development and all that has already been said about them in the discussion of the inductive type of development, applies equally as well to the deductive type and need not be repeated here.

Verification. The step of verification is necessary as a check on the reasoning process by means of which the new truth was obtained. The ultimate test of the validity of any supposed truth is to see if it tallies with observed facts, to see if it works out in experience. An hypothesis arrived at by a line of deductive thought without reference to concrete situations should always be verified by applying it to concrete situations in order to see if it really works.

Difference Between Inductive and Deductive Development. The only difference between the inductive and deductive types of development is in the method used in forming the hypothesis. In the deductive development, instead of examining and comparing individual cases to discover the new truth, the teacher and pupils review their stock of general knowledge to see if they already have any knowledge that may help them to solve the new difficulty. No general directions can be given for this step but its nature should be clear from the following illustrations.

A Deductive Development of Rule for Multiplying an Integer by 100

Problem. To discover a short way of multiplying an integer by 100.

Recall and Combination of Old Knowledge. The teacher first reviews with the pupils the short way of multiplying by 10, namely, by annexing one cipher at the right. The fact that 100 is equal to 10×10 is next considered and therefore that multiplying by 10 and that result by 10 is the same as multiplying by 100.

Hypothesis and Verification. From this the pupils form the hypothesis that in order to multiply by 100 they must annex two ciphers. This concluson is then verified by applying it to cases such as 100×27 and checking the result obtained by adding twenty-seven 100's, or by multiplying 100 by 27.

$$\begin{array}{r} 100 \\ \underline{27} \\ 700 \\ \underline{200} \\ 2700 \end{array}$$

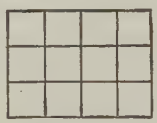
Generalization. Having verified their hypothesis, the pupils state a rule, as: "To multiply an integer by 100, annex two ciphers at the right."

Application. The rule is then applied to abstract drill examples to make sure that the pupils understand and can use it and to start fixing it in their minds. Application might also be made to concrete problems.

A Deductive Development of Rule for Area of Rectangle

Problem. To find a short way of determining the area of a rectangle.

Recall and Combination of Old Knowledge. Each pupil is supplied with several rectangular pieces of cardboard whose dimensions are all an integral number of inches, with a supply of square inches cut from cardboard, and with a foot rule. The teacher reviews the meaning of measuring and sees that the pupils know that to measure is to find how many times some standard unit is contained in the magnitude to be measured. The pupils already have had some experience in finding areas of rectangles by folding into square inches, or by fitting on square inches and counting, and this is reviewed at this time. The pupils take rectangles, 3 inches by 4 inches, and find the area by covering with square inches and counting. Their rectangles then look as follows:



Next, the teacher points out that this method of finding the area is long and inconvenient, and interests the pupils in finding a shorter method. The pupils already know how to find the number of desks in the room, or the number of postage stamps in a sheet without counting them one by one. This knowledge is reviewed and applied to the problem in hand, leading the pupils to see that "The number of square inches in the rectangle equals the number of rows times the number of square inches in one row."

Next, the possibility of finding how many rows there will be and how many square inches in each row without fitting on the square inches is considered and the pupils are led to see that, since each square inch is one inch wide, there will always be as many square inches in one row as there are inches in the base; and, since each row is one inch high there will always be as many rows as there are inches in the altitude.

Hypothesis and Verification. From the above the pupils form the hypothesis that the area of a rectangle may be found by multiplying the number of inches in the base by the number of inches in the altitude. This hypothesis is verified by taking the other rectangles, measuring the base and altitude of each with the foot rule and multiplying. The result thus obtained is then checked by fitting on square inches and counting.

Generalization. Having verified their hypothesis the pupils formulate the rule, "The area of a rectangle is equal to the product of the base and the altitude." This rule should also be stated in abbreviated form as $A=a\times b$.

Application. The application should be to actual rectangles such as the tops of the pupils' and teacher's desks, the room itself, etc. In each case the pupils should first make the proper measurements for themselves and then determine the area.

The following inductive developments of the same rules, if carefully compared with the deductive developments given above, will make clear the difference between discovery by induction and discovery by deduction, particularly if the student will remember that the fundamental difference between the two methods is that in the first the new truth is arrived at by the study and comparison of individual, concrete cases, while in the second, it is reasoned out from general truths already accepted as valid.

An Inductive Development of Rule for Multiplying an Integer by 100

Problem. Same as in the deductive development of the rule given above.

Selection of Cases.

$$100\times 7=$$
$$100\times 9=$$
$$100\times 23=$$

Comparison and Abstraction.

$$100\times 7 = 7\times 100 = 700$$
$$100\times 9 = 9\times 100 = 900$$
$$100\times 23 = 23\times 100 = 2300$$

The results in these examples might be found by any one of several different methods. The pupils might get 7×100 by thinking of it as 7×1 hundred$=7$ hundred, by adding seven 100's, or by multiplying 100 by 7.

$$\begin{array}{r} 100 \\ \underline{\times\ \ 7} \\ 700 \end{array}$$

Having obtained the results by any of these methods the teacher gets the pupils interested in finding a short way of obtaining the results. To do this they compare the result with the original number in each case and find that the result is always the original number followed by two ciphers.

Hypothesis and Verification. Same as in the deductive development given above.

Generalization. Same as in the deductive development.

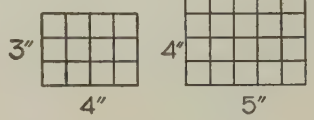
Application. Same as in the deductive development.

An Inductive Development of Rule for Area of Rectangle

Problem. Same as in the deductive development given above.

Selection of Cases. The pupils are supplied with the same material as in the deductive development.

Comparison and Abstraction. The pupils first find the areas of two or more rectangles by covering them with square inches and counting. Each pupil then has before him two or more rectangles as follows:



Again the question of finding an easy method for determining the area is raised. In order to do this the teacher calls the pupils' attention to the altitude, base and area of each rectangle and they see that the altitude of the first is 3 inches, its base 4 inches and its area 12 square inches. In the second rectangle they see that the altitude is 4 inches, the base 5 inches and the area 20 square inches.

Hypothesis and Verification. By noting the relation between the altitude, base, and area in the two rectangles, the pupils form the same hypothesis as in the deductive development and verify it in the same way.

Generalization. Same as in deductive development.

Application. Same as in deductive development.

Combination of Inductive and Deductive Methods

The Two Methods Not Distinct. For convenience of description the inductive and deductive methods of development have been considered separately. In actual teaching the two methods are often combined so that a given development lesson may be of neither type but a combination of both. In discovering a new truth the teacher need not confine herself to the examination of individual cases nor to the use of general principles already known, but can make use of both as they best serve her purpose.

Illustrations. In developing the rule for finding the volume of a cone, the pupils may, by using hollow models and filling with sand, establish *inductively* the fact that the volume of a cone is always just one-third the volume of a cylinder having the same base and altitude. Using this fact and the rule that the volume of a cylinder is equal to the product of its base and altitude, they arrive *deductively* at the conclusion that the volume of a cone is one-third the product of its base and its altitude.

If, in the deductive development of the rule for finding the area of a rectangle, which was given above, the pupils

have difficulty in seeing that the number of square inches in a row is always the same as the number of inches in the base, they could take several rectangles with different bases and by fitting on the square inches and counting, find that if the rectangle is 4 inches long there are 4 square inches in each row; if the rectangle is 6 inches long there are 6 square inches in each row, etc. If this is done the development of this particular point is inductive but the whole lesson might still be called deductive as the main trend of thought is deductive.

In the inductive development of the rule for multiplying an integer by 100, given above, the results of the individual cases 100×7, 100×9, and 100×23 were found by using the pupils' previous knowledge of multiplication. This part of the development was therefore *deductive*. The final hypothesis (the rule for multiplying an integer by 100) was discovered by comparing the individual cases, that is, by *induction*. Such a development, although partly deductive, is usually called inductive because the hypothesis is obtained by induction.

Objective Development May Be Purely Inductive. The only purely inductive developments in arithmetic are those in the very beginning in which objective situations are studied and compared and in which the hypothesis is formed from the objective situations themselves, without the aid of general knowledge already in the pupils' possession.

Not Necessary to Distinguish Between Two Methods. In actual teaching the two methods are usually so interwoven that it is often impossible to say whether a given development is more inductive or deductive, and it is not at all necessary to do so as the teacher in discovering new knowledge has at her disposal both the method of examining and comparing individual cases and the method of applying general truths already accepted, and can use

either or both as best serves her purpose. In other respects, as has been shown, the two methods of development are identical so no distinction need be made.

The Advantages and Limitations of Each Method

The teacher should keep in mind, however, the following with respect to the two methods:

I. In general the inductive method of development is the easier for the pupils because it is more concrete. The deductive method is more difficult because it is more abstract. For this reason the purely deductive method is seldom used in the grades except in developing the simplest facts.

II. Inductive discovery is possible from the first grade, while the deductive method is not possible until the pupils have accumulated a store of general knowledge in arithmetic.

III. The deductive method is usually shorter and therefore more economical of time. For this reason it should be used whenever it is not too difficult for the pupils.

IV. Inductive development is incomplete in itself; as has already been pointed out, it serves only to discover what *seems* to be true. It does not prove that the apparent truth holds in all cases. A certain relationship might hold in ninety-nine cases and fail in the hundredth. Further, pure induction shows *what* is apparently true but does not show *why* it is true. It discovers isolated bits of knowledge but does not show the relation of the new knowledge to other knowledge that the pupils already have in their possession.

Deductive Explanation

Definition. A truth known only as a separate, isolated truth is only half known. To be fully known it must be fitted into its proper place in connection with other truths

so as to form an organized body of knowledge. By *induction* one can discover that multiplying both the numerator and denominator of a fraction by the same number does not change the value of the fraction; by *deduction* one can see that this truth is not just a happen-so but the necessary and direct consequence of our previously accepted definitions and principles. The new truth is explained when it is fitted in with and seen to be the consequence of known truths, and it is in this sense that the term *explanation* is used throughout this book.

When to Explain. Truths discovered by induction must later be explained by deduction. In some cases the explanation follows immediately upon the discovery, in other cases it is postponed for several years. The ideal time to give an explanation is when the pupils see the need for it and become interested in the why of the rule or process. In the beginning, the pupils are chiefly interested in finding out what is true but later they also become interested in seeing why things are true. Just when this interest in the why of things begins, no one can say, as it varies in different individuals. Certainly no deductive explanation should be attempted in the primary and lower intermediate grades; the problem there is to lead the pupils to discover truth.

Even in the lower grades, it is true, the pupils are constantly asking "Why?", and we often mistake this for an interest in reasons and causes whereas it is usually only an interest in discovering *what* is true. The following incident illustrates the point. A child and his father were standing on the platform in front of a railroad station watching some sparrows. The boy asked his father for one of the sparrows to play with and the father told him he could not have it. The child immediately asked "Why?" and the following dialogue ensued:

"Because I can't catch it."

"Why?"
"Because I can't fly?"
"Why?"
"Because I haven't any wings."
"Why?"
"Because God did not give me any."
"Why?"

This was too much for the father, so he threatened to punish the boy if he did not stop asking so many questions. How much better it would have been if the father, when his son asked him for a sparrow, had suggested that they try to catch one. How long would it have taken to thoroughly convince the boy that he could not have a sparrow with which to play? The boy's "Why?" was not a demand for a reason but simply meant that he was not yet convinced.

In the intermediate and in the grammar grades, some of the brighter pupils become interested in the why and are perfectly capable of following the deductive thinking involved in the explanation of some of the more simple truths of arithmetic which they have previously discovered by induction. Thus in reviewing fractions in the eighth grade the pupils, although they have known for years that multiplying both terms of a fraction by the same number does not change the value of the fraction, are interested in considering why this is true.

The Teacher's Function. The teacher should not thrust explanations on the pupils, but should do all in her power to arouse and stimulate the pupils' interest in knowing the why of things and when the question arises should be ready to give the explanation or help the pupils find it for themselves. This should not be taken to mean that long formal explanations should be demanded of the pupils. Even in the upper grades, the pupils may not be able to reason out and give formal explanations for themselves

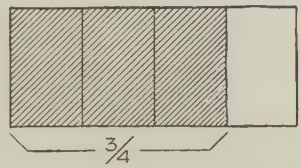
but they are entirely capable of seeing relationships when they are pointed out by the teacher.

Illustration. As an illustration of this type of deductive work, let us consider the rule for multiplying an integer by ten. Suppose this rule has been worked out inductively in the fourth grade and it is being reviewed in the sixth grade as a preparation for extending to the case of multiplying a decimal by 10. The pupils have stated the rule, "To multiply a number by ten annex a cipher to the right," and the question has arisen "Why does annexing a cipher multiply a number by 10?" To help the pupils discover the answer to this question, the teacher writes on the board the number 384 and, by questioning the pupils about the place value of each digit both before and after annexing the cipher, brings out the explanation, namely, that annexing a cipher multiplies a number by ten because it pushes each digit one place to the left and into a place whose value is ten times as great as that of the place previously occupied by the digit.

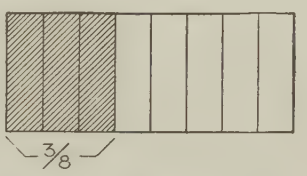
Steps in Deductive Explanation. This illustration shows the stages or phases of this kind of deductive thinking: (*a*) The problem, some general truth to be verified or explained, (*b*) The recall and selection, under the guidance of the teacher, of the general truth or truths that explain or prove the given truth, and (*c*) The formulation of the explanation. These stages do not represent separate and distinct steps that must be taken in that particular order but rather different phases of the thinking that must be done in order to arrive at the explanation.

Illustrations. As another illustration of this sort of work suppose the question has arisen, "Why does multiplying the denominator of a fraction divide the value of the fraction?" The teacher might help the pupils form their own explanation by questions such as, "In the fraction ¾ what does the denominator, 4, tell?" (The number

of equal parts into which the whole has been divided.) "What does the numerator, 3, tell?" (The number of equal parts that we have in our fraction.) "Picture the fraction ¾."



"If you multiply the denominator by 2, how many parts will there be in the whole?" (8.) "Picture the resulting fraction, ⅜."



"Are there more or fewer parts in the whole than before?" (More, just twice as many.) "Are they larger or smaller?" (Just half as large.) "So multiplying the denominator by 2 did what to the fraction?" (Made it just half as large, or divided it by 2.) "Why?" (Because it made twice as many parts in the whole and so they were just half as large.)

As a final illustration suppose the pupils have just discovered the rule for finding the area of a rectangle, and have discovered it by the inductive method given above. The question might arise as to why the rule works. The teacher could then show the pupils that the number of inches in the altitude is the same as the number of rows of square inches and that the number of inches in the base is the same as the number of square inches in one

row, so that when they multiply the number of inches in the base by the number of inches in the altitude they are really multiplying the number of square inches in one row by the number of rows and so the result obtained represents the number of square inches in the rectangle, that is, the area of the rectangle.

CHAPTER V

THE DEVELOPMENT OF NEW IDEAS

The Basic Ideas of Arithmetic. One of the most important tasks of the teacher, particularly in the primary grades, is to see that the pupils get certain ideas that are fundamental to the work in arithmetic. The most important of these are (*a*) Number Ideas, (*b*) Fraction Ideas, (*c*) Operation Ideas, (*d*) Geometric Ideas, and (*e*) Measuring Ideas. This side of the teaching of arithmetic has been badly neglected in the past. Formerly, the symbol as 3 or ⅔ was given first and it was thought that the pupils would get the idea from the symbol. Today the teacher first sees that the pupils get the idea through concrete experience and then gives the symbol. Too often in the past the pupils have been dealing with meaningless symbols instead of with ideas. The author was once asked to help a boy in the fifth grade who had become completely lost and bewildered in the work with fractions. It only took a few minutes to locate the difficulty, the boy did not know the meaning of fraction; two-thirds called to his mind nothing but the symbol ⅔. After a few days spent on concrete, objective representation of fractions, the boy had acquired clear fraction ideas and in a few more days caught up with his class and had no further difficulty.

Number Ideas

The pupils when they enter school usually have certain number concepts. These, however, are apt to be rather vague and uncertain and there is usually great variation between individual pupils. It is the duty of the teacher in the primary grades to take this knowledge of the pupils

and extend and clarify it so as to form a firm and dependable basis for the future work in numbers.

Number Idea Complex. The number idea is very complex and is not acquired all at one time but gradually and as the result of wide and varied experience. The pupils' first ideas are of necessity incomplete and vague and must be added to and clarified by further experience. Thus the complete concept of six involves the notion of position in a series, as the sixth pupil in a row, or the sixth house from the corner; the idea of the number in a group, as six pencils or six boys; the idea of magnitude, as a line six inches long or an object that weighs six pounds; the knowledge that six is one more than five and one less than seven, two more than four and two less than eight, etc.; and the knowledge that six is made up of two threes or three twos.

Order and Method of Developing. The best order and the best means of developing these various number ideas has long been a matter of dispute. This question will not be entered into here, as we have no convincing evidence on either side and the whole question is probably much less important than it has usually been considered. The important thing is that the pupils gradually develop a comprehensive understanding of six and of each of the other numbers and as long as they do this the means used and the particular order in which the elements of the complex idea are grasped are matters of comparatively little importance. The following scheme for developing the number ideas is not presented as the only method but simply as one possible method.

Number as a Sequence. The pupils first get the idea of number as meaning position in a series. This is obtained through rote counting. This counting serves to fix in the pupils' minds the sequence of the numbers. Much practice should be given in counting objects of various sorts with-

out any attempt at first to have the pupils recognize any of these numbers of things. Number rhymes such as the following can be used to advantage in this connection:

"One, two, buckle my shoe,
Three, four, bolt the door,
Five, six, pick up sticks,
Seven, eight, lay them straight,
Nine, ten, a good fat hen."

"One, two, three, four, five,
I caught a hare alive,
Six, seven, eight, nine, ten,
I let him go again."

Fortunately children like to count for the sake of counting and this natural liking should be taken advantage of in teaching numbers. This kind of counting is called *rote counting* and should go considerably beyond the pupils' present number needs.

Number as a Measure. Having become acquainted with the sequence of the numbers from rote counting the pupils must get the idea of number as answering the questions "How many?" and "How much?" They can do this by counting groups of objects and by measuring. Thus the "how many" idea of six is obtained by counting various groups of six objects each, as six apples, six boys, six girls, six hair-ribbons, etc.; and by doing things six times, as tapping a bell, clapping the hands, knocking on the door, taking steps, etc. Counting of this kind is called *rational counting.*

Through their experience in the rational counting of objects the pupils should get such a definite idea of two, three, and four that they can recognize groups of two, three, and four objects at sight without stopping to

count, and can recognize larger groups by breaking them up into groups of twos, threes, and fours. Thus a group of seven objects should be recognized without counting, by mentally grouping into three and four. Some pupils can recognize five or, in some instances, even a larger number of objects without counting, but four seems to be the maximum for most pupils.

The "how much" idea is developed through measuring and the pupils should be given much practice in this kind of work, using all of the common standard units of measure. Thus the pupils get the "how much" idea of six by measuring lines six inches long, measuring out six pints of water, etc.

Rational counting and measuring are both necessary to the development of complete number ideas and should be carried on together, starting as soon as the sequence of the first ten numbers has been memorized through rote counting. Through these two activities the number names and symbols for the smaller numbers become something more than mere words and symbols to the pupils and gradually come to stand for definite ideas.

Just how far the pupils can get clear ideas of the numbers represented by the words and symbols is a question difficult to answer. Before they leave school 100 should undoubtedly stand for a distinct conception and possibly the same is true of 1000. Most adults probably have no definite conception of numbers above a thousand.

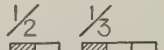
Learning Each Number in Its Relation to Other Numbers. A number is not completely known when it is known as standing for a definite place in the sequence and as answering the questions "how many?" and "how much?" The pupils may be able to count up to and beyond six, and may also be able to recognize and form groups of six objects, and to repeat simple acts six times, but they do not have the complete idea "six" until they know six

in all of its relations to the other numbers. They already know from rote counting that six comes after five and before seven, and they probably know from rational counting and measuring that six is one more than five and one less than seven. They must also know that six is two more than four, three more than three, two less than eight, three less than nine, etc.; and that six is made up of two 3's or three 2's. These relations can be obtained partially through rote counting by 1's, 2's, 3's, etc., but to make them stand for concrete ideas they must also be worked out by making comparisons of objects and groups of objects. Thus, the pupils find that there are three more objects in a group of nine than in a group of six, and that a line nine inches long is three inches longer than one that is just six inches long. The comparisons at first should be indefinite, as "Which group is the larger?" "Which line the longer?" etc. This should be followed by work in which the comparisons are made definitely with the aid of counting and measuring, in order to tell "how much larger?" or "how much longer?" Indefinite comparisons should start before the work in rational counting and measuring, and the making of definite comparisons should grow out of the indefinite comparisons and should provide the motive for much of the rational counting and measuring. The ideas and many of the facts of subtraction, and multiplication grow directly out of the work in making comparisons.

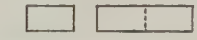
Fraction Ideas

Fraction Idea Complex. The fraction idea, like the number idea, is a complex one and is a gradual growth as a result of the pupils' experience. The following outline shows approximately the stages in the development of the complete fraction idea.

I. Unit Fractions. ½, ⅓, ¼, etc.

(*a*) Part of a single object, as 

(*b*) Part of a group of objects, as 0 0 0 0 0 0 (½)

(*c*) Comparison of two single objects, as: pint=½ of a quart; foot=⅓ of a yard; one object ½ as large as another, as 

(*d*) Comparison of two groups, 0 0 0 is ½ as many as 0 0 0 0 0 0.

All four of these ideas of the simple unit fractions should be started as early as possible—in the first grade if any number work is done there. The order in which the pupils obtain these four ideas of ½ is of no importance. Some teachers start with the comparison idea, using the pint and the quart as the first introduction to the fraction idea. Most teachers prefer to start with the part idea, regarding it as more simple. When this is done the pupils' experience in sharing single objects and groups of objects with others can be made the introduction to fractions. The important thing is that the pupils in their concrete experiences with ½ meet all four of the ideas; ½ of an object, ½ of a group, one object ½ as large as another, and one group containing ½ as many objects as another group.

II. Simple Proper Fractions. Proper fractions that are not unit fractions, as: ⅔, ¾, ¾.

(*a*) Part of a single object, as

(*b*) Part of a group, as 0 0 0 | 0 0 0 | 0 0 0 (⅔)

(*c*) Comparison of two objects, as
The lower rectangle is ⅔ of the upper.

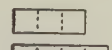
(*d*) Comparison of two groups. 000 000 000 / 000 000 6 is ⅔ of 9. This work should start early in the third year.

III. Improper Fractions. $\frac{3}{2}$, $\frac{5}{4}$, etc. Improper fractions are first met late in the third or early in the fourth grade and arise from the work in addition of fractions, as $\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$. When encountered in this way their meaning should be made clear by picturing them objectively, using both single objects and groups as the unit.

(*a*) Part of a single object.

(*b*) Part of a group. 00 00 00 / 00 00 00 $=\frac{4}{3}$ of 6.

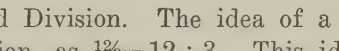
Later the idea of one object being $\frac{4}{3}$ or $1\frac{1}{3}$ times as large as another object, and one group or number being $\frac{4}{3}$ or $1\frac{1}{3}$ times as large as another group or number must be introduced. If necessary, these ideas may also be objectified.

(*c*) Comparison of two objects. 

The lower rectangle is $\frac{4}{3}$ of the upper.

(*d*) Comparison of two groups. 00 00 00 / 00 00 00 00. Eight is $\frac{4}{3}$ of six.

IV. Indicated Division. The idea of a fraction as an indicated division, as $1\frac{2}{3}=12\div3$. This idea is first met in connection with reducing an improper fraction to a mixed number. The pupils already know that $1\frac{2}{3}=4$, or if they do not they can picture $1\frac{2}{3}$ as



They also know that $12\div3=4$. Therefore it is evident that $1\frac{2}{3}=12\div3$. From this and similar cases the pupils see that a fraction is an indicated division.

Operation Ideas

One of the most important purposes of the work in the primary grades is to give the pupils clear, usable ideas of the four fundamental operations of addition, subtraction, multiplication, and division. These ideas are obtained through the work with objective situations, already discussed, and through the solution of concrete problems involving these operations. The test of whether the pupils have clear ideas of these operations is their ability to recognize addition, subtraction, multiplication and division problems.

The Addition Idea. The addition idea is comparatively simple. Through the work with objects and problems the pupils come to think of addition as a putting together or combining process. No formal definition of any of the processes should be attempted in the grades.

The Subtraction Idea. Mathematically, subtraction is simply the inverse of addition. It is the process by which the second addend is found when the sum of two addends and one of the addends is known. This idea of subtraction is too abstract for pupils in the grades. The important thing for them is that they be able to recognize the various types of problems that are solved by means of subtraction. There are three of these types.

(*a*) The Remainder or Take Away type.

"If Johnny has seven pieces of candy and eats two, how many does he have left?"

(*b*) The Difference or Comparison idea.

"Mary has 7 apples and Julia has 2. How many more apples does Mary have than Julia?"

(*c*) The Additive idea.

"Johnny sees a toy in a shop window marked 10¢. If he has 7¢, how many more must he have to buy the toy?"

If the pupils are taught to subtract by the take-away method their fundamental idea of subtraction will be the

take-away idea, but they must know that subtraction is also used to find how much larger one number is than another, and to find how much must be added to a smaller number to make a larger, as well as to find what is left when part is taken away. They get this knowledge from meeting plenty of subtraction problems of all three types. On the other hand, if the pupils are taught to subtract by the additive or Austrian method they look upon subtraction as the process of adding to a smaller number to get a larger, but again they must learn from experience with problems of all three types that subtraction is used in the two other types of problems. To avoid confusion the objective work should be carried out according to the particular plan to be used in subtraction. That is, in working out the subtraction fact $\begin{array}{r} 7 \\ -3 \\ \hline \end{array}$ the pupils should start with seven objects, take away three and count the remainder if they are to subtract by the take-away method. But if they are to subtract by the additive method they should start with three objects and add enough to make seven, counting the number added. As soon as the pupils have passed the objective stage they must meet problems of all three types.

The Multiplication Idea. Pupils gain their idea of multiplication from working with objects and from concrete problems. They regard multiplication as the "times process" or the process of finding what three 2's or seven 8's equals without adding. Again no formal abstract definition should be attempted. The test of the pupils' grasp of the multiplication idea is their ability to recognize concrete multiplication problems and to distinguish them from addition, subtraction, and division problems.

The Division Idea. Mathematically, division is the inverse of multiplication. It is the process in which the product of two factors and one of the factors is known,

to find the other factor. Thus two cases arise depending on whether the known factor is the multiplier or the multiplicand. In the example $3 \times \$4 = \$12$, 3 tells the number of parts, \$4 the number of dollars in each part, and \$12 the number of dollars in the whole. Corresponding to this multiplication situation there are two inverse division situations, namely, $\$12 \div \$4 = ?$, and $\$12 \div 3 = ?$ In the first of these the number of dollars in the whole and the number of dollars in each part are known to find the number of parts. Briefly, the question is "How many \$4's are there in \$12?" In the second the number of dollars in the whole and the number of equal parts are known to find the number of dollars in each part. Briefly, the question is "If \$12 is divided into 3 equal parts, how many dollars are there in each part?" Evidently another way of stating this question is "⅓ of $\$12 = ?$"

The pupils' first and fundamental idea of division is based on the first situation given above, that is, the pupils think of division primarily as "finding how many times one number is contained in another." The objective work in division is all based on this idea. At the same time, however, that the pupils are finding objectively how many 2's there are in 8, etc., and calling the process division, they are also dividing 8 into 4 equal parts and finding ¼ of 8. Later the pupils learn that the short way to find ¼ of a number is to divide the number by 4. So the pupils not only think of division as "finding how many times one number is contained in another" but also know that division is used to find parts of numbers. Plenty of concrete division problems of both kinds must be met and solved by the pupils.

Geometric Ideas

The pupils before leaving the grades should know the names of and be able to recognize the common geometric

forms, such as the square, rectangle (oblong), triangle, circle, cube, and sphere. The important thing is that the pupils be able to recognize these figures, not that they learn formal definitions.

Measuring Ideas

Another group of ideas fundamental to the arithmetic work in the grades is the group connected with measuring. These are (*a*) the idea of measuring and (*b*) the ideas of the standard units of measure.

The Idea of Measuring. The measuring idea is the natural outgrowth of the work in making comparisons. As previously stated the first comparisons should be indefinite, simply "this is larger than that," "this is shorter than that," etc. Later the desire to make more accurate comparisons, to tell "how much larger" and "how much shorter," makes it necessary to introduce the idea of measuring a magnitude by comparing it to another known magnitude of the same kind that is used as a unit of measure. At first the units used in measuring should be indefinite units such as the pace, the length of the finger, etc. Later the standard units should be introduced.

The Ideas of the Standard Units. It is not enough that the pupils know the names of the standard units and the relations between them (Tables); the important thing is that they have a clear idea of the more common units such as the inch, the foot, the pound, etc. These can only be obtained by much practice in measuring, using these units, and in estimating quantities and then verifying the estimate by measuring. The test of whether the pupils know what an inch is or not, is their ability to estimate lengths. If they guess that a three-inch line is five inches long they evidently have no very accurate idea of an inch, but by continued estimating and verifying by measuring they will gradually get a correct idea.

The Development of New Ideas

Illustration. As has already been stated the teacher can not convey a new idea such as the idea "square," "one-half," or "7" to the pupils through words; the pupils can only grasp the idea through their own experience. To make clear the process by which a new general idea or concept is acquired let us consider how a baby gets the concept "dog." In the first place the child must see not one dog but several dogs and these dogs must not be all alike in any way except in that they are all dogs. If the dogs seen are all white the child will think that whiteness is a necessary attribute of dogs and will fail to recognize a black dog. Seeing a variety of dogs is not enough however. Each time some adult must be present to give the name dog. The formation of a general idea is a process of abstraction. At first the child connects the name dog with an individual dog, say a small, white, curly dog; then a large, black, short-haired animal is pointed out to him and called dog and so by seeing dogs of all kinds and having the name dog given him each time, the child gradually comes to associate the word dog not with any particular dog but with those attributes that are common to all dogs.

As soon as the baby gets an imperfect concept dog he will probably try to use it. He will point to a small four-legged animal and say "Dog." The animal happens, however, to be a cat and the baby is corrected by some adult. In this way, through use, the vague, imperfect idea is gradually clarified and perfected and becomes a permanent possession of the baby's mind. From this illustration it is seen that the acquisition of a new idea is essentially an inductive process and goes through the same phases as all other inductive thinking.

The Steps Can Not be Separated. The different phases

of inductive thinking are never separate and distinct and this is particularly true in the development of an idea. The phases are usually so merged together that no line can be drawn between them. Indeed they are usually carried on practically at the same time. As each case is studied, attention is directed towards the essential elements and a name or other symbol is connected with it; and as soon as the idea is partially grasped, the pupil commences to use it and through use verifies and fixes the idea. Each phase of thought is, however, present, and to be unaware of, or neglect any of these phases, is to invite failure.

The inductive development of an idea often differs from the development of facts or processes in still another way. In the latter the induction is *conscious,* in the former it is often *unconscious.*

Selection of Cases. The teacher must remember two things in this connection. (1) A general idea can not be obtained from a single instance. Several situations having one element in common must be presented. (2) These situations must be different in all ways except in respect to the one common element.

Comparison and Abstraction. As the pupils meet each situation their attention must be directed toward the essential element, or elements, common to all the situations. The pupils can not get the idea square by looking at square objects if they are thinking about their size or color; they must be made to think about their shape. The most important ways of directing the attention toward the required element are by using situations in which the element is just as prominent as possible, and by using a variety of situations having only the one element that is common to all of them. This one element will stand out from the others because it is the one thing present in all. Sometimes attention can be directed towards the common elements by leading the pupils to compare the different

situations, eliminate the varying elements and pick out those that are common. Thus in the case of the square the pupils might be shown several squares of different sizes, colors and materials. They might compare and find these differences and then look for likenesses, finding that they all have four corners, and four sides and that the sides are all equal. Finally, the pupils' attention can sometimes be directed towards the desired elements by reviewing other similar ideas that they already have. In developing the idea of 7 the pupils can be made to think about number and not size, shape, color, or something else by reviewing briefly the numbers that they already know.

Generalization. In the case of an idea, the step of generalization can not be separated from the other steps. The generalization is made more or less unconsciously as the various individual cases are met. To aid the pupils in unconsciously generalizing, a word or other symbol must be given with each of the individuals or situations as it is presented, so that eventually it will come to be connected in the pupils' minds not with any one of the different situations but only with the elements common to all.

Further, the generalization can not always be formally made in words. It would be extremely difficult, for instance, to state clearly in words the common elements that go to make up a dog and distinguish dogs from cats and other animals of the same general size and shape. In the case of arithmetical ideas the pupils can generalize in words their idea of square or of ½ but cannot very easily put in words their idea of 9.

Verification and Application. The test of the correctness of an idea is the ability to use and apply it successfully. The best test of whether the pupils have correct ideas of measuring and of square is not their ability to define these in words but their ability to measure successfully and to recognize squares. Thus the Verification is part of the

Application, in the case of ideas, and the latter step is therefore doubly important. As soon as the pupils have partially grasped the new idea they must be given many and varied opportunities to apply it and by so doing to correct their idea and to make it their permanent possession.

CHAPTER VI

THE DEVELOPMENT OF FACTS AND PRINCIPLES

Facts. There are a large number of facts to be presented in arithmetic. The simplest of these are the fundamental facts of addition, subtraction, multiplication and division—the so-called combinations and tables. Besides these, there are other facts or relations such as the fact that the circumference of a circle is always a little more than three times the diameter, the fact that the volume of a cone is just one-third the volume of a cylinder having the same base and altitude, the fact that a number is divisible by 9 if the sum of its digits is divisible by 9 and not otherwise, etc.

Principles. Besides these facts, there are truths of a more general and fundamental character, called principles, such as the principle that a series of factors may be multiplied in any order without changing the result, and the principle that multiplying both terms of a fraction by the same number does not change its value. No hard and fast distinction can be made between facts and principles and it is not necessary to do so.

Sometimes a general principle is best presented in the grades as a rule or set of directions for accomplishing a particular task. Thus the general principle that multiplying the denominator of a fraction by any number, divides the fraction by that number, instead of being presented as a general principle, might be presented as a rule for dividing a fraction by an integer,—namely "To divide a fraction by an integer multiply the denominator by that integer." This has the advantage of closely connecting the principle and its most important application.

Addition Facts

Number of Addition Facts. In addition the pupils must know the sum of any two numbers from 0 to 9. In all, there are one hundred of these sums. Thirty-six of these are reverses such as $3+4=7$, the reverse of $4+3=7$; and nineteen more have zero as one of the two addends, as $0+0=0$, and $0+3=3$. Eliminating these, there are forty-five different addition facts remaining, that are usually spoken of as the forty-five addition combinations.

The thirty-six reverses can not, however, be eliminated entirely. If the pupils first learn that $3+5=8$ and that $2+4=6$ and later learn that $5+3=8$ and $4+2=6$, they should be led to generalize from these and similar facts to the effect that the two addends may be interchanged without changing the result. This generalization will save them much time in learning new facts for if they know that $7+9=16$ they also know that $9+7=16$. They will then need some drill on the latter fact so that they will know the sum instantly without having to stop and mentally interchange the two numbers. The zero combinations should also be taught, as they are needed in examples such as $\begin{array}{r} 30 \\ \underline{27} \end{array}$. The idea of adding zero, or nothing, is more difficult than the idea of adding one or two, as it involves a more general and abstract conception of the addition process. The addition facts involving zero, however, once understood, are easily remembered so less time need be spent drilling on them than on the other facts. It follows from the above discussion that there are not forty-five but one hundred addition facts to be taught and memorized. Some of these are much easier than others, and some are connected with others in such a way that knowing one fact the other fact can easily be obtained and memorized, but all of the hundred must be taught and committed to memory.

Order of Teaching. The order in which the addition facts are first presented differs in different schools. Some courses of study organize these facts in tables and present the tables in order, as follows:

1	2	3	4	5	6	7	8	9
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$
1	2	3	4	5	6	7	8	9
$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	$\frac{2}{9}$	$\frac{2}{10}$	$\frac{2}{11}$
1	2	3	4	5	6	7	8	9
$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{3}{9}$	$\frac{3}{10}$	$\frac{3}{11}$	$\frac{3}{12}$

These are known as the tables of 1's, 2's and 3's, respectively. There are similar tables of 4's, 5's, 6's, 7's, 8's and 9's. In many schools the combinations having sums of 10 or less are taught first, and those having sums greater than 10 are postponed until later. If this is done and the above organization is followed, the combinations having sums greater than 10, such as $\frac{9}{2}$, $\frac{8}{3}$ and $\frac{9}{3}$ are omitted from the tables when first presented and taught later when the table is being reviewed.

If the addition facts are taught in tables, as above, each table after the first will contain some facts that are the reverses of facts already known. Thus, in the table of 2's the fact $\frac{1}{2}$ occurs, which is the reverse of $\frac{2}{1}$ which occurs in

the table of 1's; and in the table of 3's are the facts $\begin{array}{r}1\\3\\\hline 4\end{array}$ and $\begin{array}{r}2\\3\\\hline 5\end{array}$ which are the reverses of the facts $\begin{array}{r}3\\1\\\hline 4\end{array}$ and $\begin{array}{r}3\\2\\\hline 5\end{array}$ in the tables of 1's and 2's, respectively. The pupils should be led to discover and make use of these relations as soon as possible so that when they take up a new table they will already know part of it but in a different form and will need only enough drill to enable them to recognize these facts in their new form.

If this organization is followed, the zero combinations, which would logically come first, should be postponed until later. One of the most important things that the pupils should get from their work with the first addition facts is a clear idea of addition. This idea can not be obtained as well from $\begin{array}{r}4\\0\\\hline\end{array}$ as it can from $\begin{array}{r}4\\1\\\hline\end{array}$ or $\begin{array}{r}4\\2\\\hline\end{array}$, as the last two can be worked out objectively and the zero combinations can not. Further, as already stated, the addition of zero is a more difficult idea to grasp than the addition of one or two. For these reasons the zero combinations should not be presented until after the pupils have met enough other combinations to give them a clear idea of addition.

Other schools make the sum the basis of the organization, teaching, in order, the combinations having sums 2, 3, 4, 5, etc., as follows:

1	2	1	3	1	2	4	1	3	2		
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{2}{4}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{3}{5}$		

In this organization the fact and its reverse are presented at the same time, when the pupils learn that $\begin{array}{r}3\\1\\\hline 4\end{array}$ they also

learn that $\begin{array}{r} 1 \\ 3 \\ \hline 4 \end{array}$. Again the zero combinations should not be taught first but introduced later. The idea of adding zero once grasped, the zero combination can then be worked out with the others, as $\begin{array}{rrrrrrr} 5 & 1 & 4 & 2 & 3 & 6 & 0 \\ \hline 1 & 5 & 2 & 4 & 3 & 0 & 6 \\ \hline 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{array}$.

More recently the attempt has been made in some schools to teach the facts in the order of their comparative difficulty. The following table contains all of the hundred facts, those in the first row being the easiest, and those in the last the most difficult:

0	0	4	1	6	0	0	7	0	3	9	5	2	0	8	0	0	0	0	1
0	3	0	0	0	8	1	0	5	0	0	0	0	4	0	6	9	2	7	1
1	1	4	1	7	6	1	2	1	1	1	2	3	1	8	4	3	5	5	9
2	8	1	5	1	1	3	2	9	6	7	1	3	4	1	4	1	5	1	1
2	3	2	4	3	8	2	6	7	4	3	5	7	2	2	5	6	3	2	3
3	4	6	2	5	2	5	3	2	3	2	2	3	4	7	3	2	6	8	7
9	4	6	4	4	8	7	9	4	3	6	2	8	3	4	9	7	8	5	9
2	7	6	6	8	3	7	4	5	9	4	9	4	8	9	3	4	8	4	9
6	9	5	7	6	9	5	8	9	5	7	7	8	6	8	9	7	5	6	8
7	7	7	8	8	6	6	5	8	9	5	6	7	5	6	5	9	8	9	9

This organization is based upon "the difficulty of memorizing" rather than "the difficulty of understanding." The zero combinations are easy to memorize but difficult to understand. For this reason, they should not be taught first, but should be postponed until after some of the seemingly more difficult facts have been learned.

As previously stated, all of the hundred facts must be taught and memorized and the particular order in which they are presented is immaterial unless it is found that they can be taught in a certain order in less time and with less effort than in any other order. At the present time we have no conclusive experimental proof as to which is the most economical order and until we do have such proof, any discussion of the relative merits of the different organizations given above is mere theorizing.

Miscellaneous Order in Drill. The above discussion has to do only with the order in which the facts are first presented. As soon as the pupils have met a few facts, the order of these facts must be constantly changed in the drill work so that each may be learned as a separate fact and not as part of a table. No matter what organization is used in presenting the facts, it must not be adhered to in the drill on these facts.

Derived Facts. Besides the hundred addition facts discussed above, there are other facts derived from these, that the pupils must know in order to add columns. In adding the column in the margin, starting at the bottom, the pupils must know the sum of 7 and 9, of 16 and 5, of 21 and 8, and of 29 and 7. The first of these is one of the fundamental facts, but in the others the pupils must be able to mentally add a two-place number (carried in the head) to a one-place number.

7
8
5
9
7

This is known as *addition by endings,* or *carrying the fundamental addition facts to higher decades.* The usual practice, in the past, has been to postpone the teaching of these facts until after the hundred fundamental combinations have been mastered and until they are needed in column addition. At present there is a tendency to teach them at the same time as the fundamental facts, teaching and drilling on $\begin{array}{r} 17 \\ 8 \\ \hline \end{array}$ $\begin{array}{r} 27 \\ 8 \\ \hline \end{array}$ $\begin{array}{r} 37 \\ 8 \\ \hline \end{array}$, etc., at the same time

as $\begin{array}{r} 7 \\ \underline{8} \end{array}$. The claim is made that this procedure results in a saving of time.

Subtraction Facts

Number of Subtraction Facts. Every addition fact has as its inverse a corresponding subtraction fact, so there are the same number of fundamental facts in subtraction as in addition, namely, one hundred. This number includes the nine combinations in which the answer is zero, as $\begin{array}{r} 1\ 2 \\ \underline{1}\ \underline{2} \\ 0\ 0 \end{array}$, etc.; the nine in which the subtrahend is zero, as $\begin{array}{r} 1\ 2 \\ \underline{0}\ \underline{0} \\ 1\ 2 \end{array}$ etc.; and also $\begin{array}{r} 0 \\ \underline{0} \\ 0 \end{array}$. These nineteen facts are easier than the other eighty-one and consequently need not be drilled on nearly as much, but can not be neglected entirely.

Order of Teaching. The usual practice today is to teach the inverse subtraction facts at the same time as the corresponding addition facts, teaching together the four facts, $\begin{array}{r} 9 \\ \underline{+7} \\ \end{array}$ $\begin{array}{r} 7 \\ \underline{+9} \\ \end{array}$ $\begin{array}{r} 16 \\ \underline{-7} \\ \end{array}$ and $\begin{array}{r} 16 \\ \underline{-9} \\ \end{array}$. It is claimed that this results in an economy of time, and while we have no exact experimental proof of this, experience would seem to indicate that this claim is true. The order of presenting the subtraction facts, therefore, depends on the order used in presenting the addition facts.

Multiplication Facts

Number of Multiplication Facts. If the multiplication facts are taught to 10×10 there are altogether one hundred twenty-one facts. This number includes twenty-one zero combinations, as $\begin{array}{r} 0\ 1\ 0\ 2\ 0 \\ \underline{0}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{2} \\ \end{array}$, etc.; nineteen other combi-

nations in which one factor is unity, as $\begin{array}{r}1\\1\\\hline 1\end{array}$ $\begin{array}{r}2\\1\\\hline 2\end{array}$ $\begin{array}{r}1\\2\\\hline 2\end{array}$, etc.; seven-

teen others in which one factor is ten, as $\begin{array}{r}10\\2\\\hline 2\end{array}$ $\begin{array}{r}2\\10\\\hline 10\end{array}$ $\begin{array}{r}10\\3\\\hline 3\end{array}$ $\begin{array}{r}3\\10\\\hline 10\end{array}$, etc.;

and twenty-eight others that are reverses, as $8\times7=56$, the reverse of $7\times8=56$, etc. This leaves thirty-six combinations on which to spend most of the time, but the easier combinations should not be neglected entirely. If the pupils know $7\times8=56$, they also know $8\times7=56$, but they should be drilled on both facts so that they can give either without hesitation. The zero facts must also receive some attention; it is a common mistake for high school or even college students to say that $0\times5=5$ or $5\times0=5$, not because they do not know better if they stop to think, but simply because these facts have never been memorized and made mechanical as have the other facts, such as $5\times7=35$.

Some schools teach the facts to 12×12. The only advantage in doing this is that it enables the pupils to multiply by 11 and 12 in a single multiplication without breaking the multiplier into two parts, and to divide by 11 and 12, using short division instead of long. It is doubtful if the time thus saved counterbalances the extra time necessary to learn the additional facts, particularly as 12 and 11 are easy multipliers and divisors anyway, and do not occur very frequently. On the other hand, it is true that the extra time taken to learn the additional combinations is not very great. The present tendency seems to be to teach the facts only to 10×10.

Order of Teaching. The custom has prevailed in the past of teaching the multiplication facts in the form of tables and teaching the tables in the order of the digits. The table of 1's is first taught, then the 2's, the 3's, etc., as follows:

1× 1= 1	2× 1= 2		10× 1= 10
1× 2= 2	2× 2= 4		10× 2= 20
1× 3= 3	2× 3= 6		10× 3= 30
* * *	* * *	etc., to	* * *
1× 9= 9	2× 9=18		10× 9= 90
1×10=10	2×10=20		10×10=100

Some schools prefer organizing the tables in this form:

1×1= 1	1×2= 2	
2×1= 2	2×2= 4	
3×1= 3	3×2= 6	
* * *	* * *	etc.
9×1= 9	9×2=18	
10×1=10	10×2=20	

Most schools teach the table of 2's first, postponing the 1's until later, as they do not bring out the multiplication idea as well as the 2's and therefore do not make as good an introduction to the multiplication facts. After the idea of multiplication is grasped the 1's will be very easy and but little drill need be given on them. As in addition and subtraction, the zero combinations should be postponed until after the pupils have a clear idea of the meaning of multiplication.

More recently many schools have broken away from these traditional orders. Such schools have usually followed one of two plans; they have either presented the tables in some order other than the regular or 1, 2, 3 order, or else have presented the facts as individual facts, not organized in tables at all. Where the tables are not presented in regular order the attempt has been made to teach them in the order of their comparative difficulty. It seems to be pretty well agreed that the four easiest tables are the 2's, 4's, 5's and 10's; as to the other tables there is a wide difference of opinion.

We have no conclusive experimental proof as to what is the most economical order in which to teach the facts or as to the comparative difficulty of the different tables and facts. Jessup and Coffman* secured the opinions of 507 superintendents on this question. Their replies are presented in the following table:

1, 2, 3, etc., Order	37.5%
2, 4, 5, 10, etc., Order	25.4%
No Tables but Combinations	6.4%
Not Important	7.3%
Miscellaneous	23.4%

That is, of 507 superintendents giving opinions, 37.5% believe that the tables should be taught in regular order; 25.4% believe they should be taught in the 2, 4, 5, 10 order; 6.4% believe that the combinations should be taught without any reference to order; 7.3% believe that the order is of no importance whatever; and 23.4% believe that any miscellaneous order will be satisfactory. Jessup and Coffman state: "The only conclusion to which we can come is that the old order is still the prevailing order, and that the prevailing tendency is to try to find some other."

Miscellaneous Order in Drill. As stated in discussing the order of presenting the addition facts, the order in which the facts are first presented is only a question of economy of time, and no matter what organization is followed, as soon as the pupils have acquired a few facts they must be drilled upon them in a miscellaneous and constantly changing order. If this is not done there is danger that the facts will not be learned as individual facts but as part of a table or of some particular organization.

*Jessup, W. A., and Coffman, L. D.—The Supervision of Arithmetic.

Division Facts

Number of Division Facts. Since for every multiplication fact there is an inverse division fact, one would expect to find the same number of division as multiplication facts, that is, one hundred twenty-one. This would be true were it not for the fact that the inverses of eleven of the multiplication facts take the form $0\overline{)0}$, $0\overline{)1}$, which is indeterminate, and eleven others have unity for the divisor, leaving only ninety-nine facts to be taught. Of these, nine have zero for the dividend, as $2\overline{)0}$, $3\overline{)0}$, etc.; nine have unity for the quotient, as $2\overline{)2}$, $3\overline{)3}$, etc.; nine are of the form $2\overline{)20}$, $3\overline{)30}$, etc.; and nine of the form $10\overline{)20}$, $10\overline{)30}$, etc. These thirty-six facts are easier to memorize than the remaining sixty-three and so less time need be spent on them.

Derived Facts. Besides the fundamental division facts the pupils must know other facts derived from them. In the division example $4\overline{)276}$ the pupils must know how many 4's in 27 and what remainder is left. To distinguish these derived facts from the fundamental facts the first are called the *uneven division facts* and the last the *even division facts.* All of the facts that the pupils must know in division both even and uneven are summarized in the following:

How many 2's in each number from 2 to 19 and what remainder?

How many 3's in each number from 3 to 29 and what remainder?

How many 4's in each number from 4 to 39 and what remainder?

How many 5's in each number from 5 to 49 and what remainder?

How many 6's in each number from 6 to 59 and what remainder?

How many 7's in each number from 7 to 69 and what remainder?

How many 8's in each number from 8 to 79 and what remainder?

How many 9's in each number from 9 to 89 and what remainder?

Order of Teaching. The present tendency is to teach the division facts at the same time as, or shortly after, the corresponding multiplication facts, the order of presenting the division facts being governed by the order used in presenting the multiplication facts. When the pupils learn $6\times8=48$, or that six 8's make 48, they learn the answer to the division fact $8\overline{)48}$ which asks "How many 8's in 48?" and time is probably saved by teaching and drilling on the two facts together.

Methods of Presenting New Facts and Principles

Inductive Development of Facts. New facts and principles may be presented either inductively or deductively. As pointed out in the discussion of Objective Work, the first addition, subtraction, multiplication and division facts must be presented inductively and objectively. It is not necessary or desirable, however, that all the facts in each of these series be worked out in this way. To present all these facts inductively would take too much time.

Deductive Development of Facts. After enough facts in the series have been worked out inductively so that the pupil understands what they mean and can image the situation involved, the remaining facts are best obtained from those already known, that is by deduction. The deductions involved are so easy that the pupils have no difficulty in making them. Thus, in addition, the combinations having sums of ten or less might be obtained inductively by counting objects. The remaining facts can be obtained deductively in a variety of ways. For ex-

ample, the pupils can get $\begin{array}{r} 7 \\ +4 \\ \hline \end{array}$ from $\begin{array}{r} 7 \\ +3 \\ \hline \end{array}$, which is already known by adding one to the answer. In subtraction, after the first facts have been obtained by counting, the remaining facts can be obtained from the corresponding addition facts. Knowing that $\begin{array}{r} 9 \\ +7 \\ \hline 16 \end{array}$, or that 16 is made up of seven and nine, the pupils can readily reason out $\begin{array}{r} 16 \\ -9 \\ \hline 7 \end{array}$ and $\begin{array}{r} 16 \\ -7. \\ \hline 9 \end{array}$

The later multiplication facts can also be obtained in several ways without resorting to the counting of objects. For example, 7×8 can be obtained by adding seven 8's, or by counting by 8's, or from 6×8 by adding another 8. Again, the pupils should get their first division facts by counting objects but after enough division facts have been worked out in this way to give them the idea of what division facts mean, the remaining facts can be gotten from the multiplication facts. Thus from the fact $7\times8=56$ or "Seven 8's are fifty-six" the pupil can answer the question asked by $8\overline{)56}$, namely, "How many 8's are there in fifty-six?"

The Inductive Development of Facts and Principles

The inductive development of facts and principles presents the same phases as all other inductive thinking.

The Problem. This preparatory phase of the presentation of new material is the same for all kinds of material and for both types of thinking so, in what follows, further discussion of it will be omitted.

The Selection of Cases. In this step the teacher should present or help the pupils to choose a few very simple cases in which the general fact or principle to be discovered is as prominent an element as possible. Usually two or

three cases are sufficient to enable the pupil to discover the relationship.

Comparison and Abstraction. In examining the cases chosen, the question is "What is true in this case?" "In this?" "What is true in all of these cases?" The truth or fact should be emphasized by having the pupil state it each time in terms of the particular situation. Thus the pupil works out with the objects themselves and states "2 books and 4 books are 6 books" and "2 girls and 4 girls are 6 girls," or, by measuring, discovers and states "The square of the hypotenuse of *this* triangle is equal to the sum of the squares of the other two sides." It is essential that the supposed relationship discovered be definitely stated by the pupil each time.

Hypothesis and Verification. The tentative generalization should be made *as soon as the pupils see the relationship involved.* After the pupils have counted and found that 2 books and 4 books are 6 books and 2 girls and 4 girls are 6 girls, they are usually ready to conclude that 2 and 4 are always 6 and they should be permitted to do so. To make them go ahead and work out the fact with books, girls, boys, erasers, pieces of chalk and so on, is a waste of time and may do positive harm. The purpose of studying the objective situations is to let the pupils discover for themselves that 2 and 4 are always 6, so that they will not only be convinced of the truth of the fact but also understand the meaning of the words and symbols in which the fact is expressed. These purposes are served just as soon as the pupils are ready to generalize that 2 and 4 are always 6. Once the discovery of the new fact is made it is both unnecessary and a waste of time to use additional cases before forming a *tentative* conclusion. To multiply situations after that is to cripple the pupils by keeping them in the concrete stage of development when they are ready to pass to the general, abstract stage.

In the case of the fundamental facts of addition, subtraction, multiplication and division, the generalization made here may well be final, as the facts are so obvious that verification is unnecessary. Less obvious facts and principles should be verified by examining other cases. Thus, after the pupils have found, by filling a hollow cone with sand and emptying it into a hollow cylinder having the same base and altitude, that this particular cone is just one-third of the cylinder, they should verify this relationship by using other sets of models of different sizes and proportions, the cone always having the same base and altitude as the cylinder.

Final Generalization. The generalization should be stated first in words and then, if possible, in mathematical symbols. In the illustration previously used, the pupils should first state their generalization in words "Two and four make six" and then in mathematical symbols $2+4=6$.

Application. As soon as the general fact or principle is obtained opportunities should be given the pupils to use it in a variety of situations in order to show its significance and to fix it in mind for the time. Whenever possible, the application should be to concrete applied *problems* rather than to abstract drill *examples*. Such concrete application not only helps fix the fact or principle in mind but also shows its importance; it connects, from the beginning, the fact or principle with the situation in which it is used, which is of the utmost importance if the pupils are to be able to *use* their arithmetic.

In the case of the fundamental number facts, several of which are usually developed in the same recitation, the attempt to use the facts in problems will soon convince the pupils that they do not know them well enough and that they need to memorize them. The need for abstract drill being thus established, a little can be given at the end of the recitation or it can be postponed until the next

day. As an illustration, suppose a class has just developed the first part of the multiplication table of 3's. As the results are discovered, they are written down, so that at the beginning of the step of application the following is on the board:

$$2\times3=\ 6$$
$$3\times3=\ 9$$
$$4\times3=12$$
$$5\times3=15$$
$$6\times3=18$$

As an application of these facts the pupils might have a "Three Cent Store" in which the various articles are all marked 3 cents. Each pupil goes to the store and buys two or more articles. Thus a pupil might go to the store and buy 4 pieces of candy at 3 cents each, paying the storekeeper with toy money. To do this the pupil and storekeeper must know what $4\times3¢$ amounts to and would get the answer by looking at the board. After enough of this work to make sure that the pupils understand the meaning of the facts the advantage of saving time by memorizing them should be pointed out.

The Deductive Development of Facts and Principles

The deductive development of facts and principles differs from the inductive only in the method used for discovering the new fact or principle. Instead of leading the pupils to discover the new knowledge by studying and comparing individual cases, the teacher, by skilful questioning, leads them to recall and use old knowledge in such a way as to bring out the new. Thus, if the problem is to work out 7×6, the teacher by questions such as, "Six 6's are how many?" "How many more are seven 6's?" "What will we have to do then to get seven 6's?" and "How many are seven 6's?" will lead the pupil to the conclusion that $7\times6=42$.

The Problem, Hypothesis and Verification, Final Generalization and Application are the same in the deductive as in the inductive methods of development and all that has been said about them in connection with the inductive method applies to the deductive method as well and need not be repeated here.

CHAPTER VII

THE DEVELOPMENT OF RULES AND PROCESSES

Definitions. For some of the simpler processes of arithmetic it is possible to formulate brief rules of procedure. Such are the rules for multiplying fractions, dividing by a fraction, multiplying decimals, etc. Other processes such as long division, column addition, etc., are so complicated that no simple rule can be stated. To distinguish between these the first are called *rules* and the second *processes.*

The Processes Are Complex. The four fundamental processes of addition, subtraction, multiplication, and division are not simple processes that the pupils can learn all at one time, but are very complex and must be presented gradually, one step at a time. A mastery of any one of them involves the ability to do many different things, demands not a single ability but many different abilities and habits. It is very important that all of these abilities and habits be systematically developed one by one and that none be neglected. In order to do this the various difficulties and habits involved in the complete process must be carefully separated and each must be mastered and mechanized before the next is introduced.

The following analyses show some of the difficulties to be mastered, and habits to be formed, in each of the four processes. These are arranged approximately in the order in which they are met by the pupils, but the exact order varies in different texts and schools.

Analysis of Difficulties in Addition

1. Facts or combinations.

$$\begin{array}{cc} 3 & 7 \\ \underline{5} & \underline{8} \end{array}, \text{ etc.}$$

II. Columns of three numbers each, sum less than ten.

$$\begin{array}{ccc} 3 & 2 & 4 \\ 2 & 2 & 1 \\ \underline{4} & \underline{5} & \underline{2} \end{array}$$

This is the next step after the combinations. The new difficulty presented is that of retaining the sum of the first two numbers in mind and mentally adding it to the third number. This work should start as soon as enough combinations have been learned and much of the drill on the combinations should be obtained from adding columns such as these. Some schools prefer to learn the combinations entirely through work of this kind, claiming that time is saved in this way. Most schools still prefer, however, to give some preliminary drill on the combinations.

III. Addition of two-place numbers without "carrying."

$$\begin{array}{cc} 34 & 23 \\ \underline{12} & 12 \\ & \underline{14} \end{array}$$

To add these the pupils must learn to break the numbers into their parts and to add the units and then the tens. This is the first place in which the pupils encounter this method of procedure, which is the basis of all of our methods of calculating with numbers larger than nine.

IV. Addition of two-place and one-place numbers without carrying.

$$\begin{array}{cc} 23 & 13 \\ \underline{2} & 3 \\ & \underline{21} \end{array}$$

The new habit the pupils must gain here is that of keeping the right hand column straight.

V. Zero difficulties.

$$\begin{array}{cccc} 36 & 20 & 40 & 20 \\ \underline{20} & 14 & 16 & 30 \\ & \underline{32} & \underline{20} & \underline{10} \end{array}$$

Even if the zero combinations have been drilled upon, zeros occurring in column addition usually give some trouble and enough examples must be met by the pupils to overcome this. Many schools do not give any drill on the zero combinations as such but postpone these until they are met in column addition. Some schools teach $\begin{array}{r} 20 \\ \underline{30} \end{array}$ or $\begin{array}{r} 2 \text{ tens} \\ \underline{3 \text{ tens}} \end{array}$ at the same time as $\begin{array}{r} 2 \\ \underline{3} \end{array}$ or $\begin{array}{r} 2 \text{ units} \\ \underline{3 \text{ units}} \end{array}$.

VI. Columns of three or more numbers, sum greater than ten, but involving no "addition by endings."

$$\begin{array}{ccc} 4 & 5 & 6 \\ 5 & 3 & 2 \\ \underline{2} & \underline{6} & 1 \\ & & \underline{2} \end{array}$$

The only new difficulty involved is learning what to do with the two figures obtained in the answer. The habit of placing the units figure of the answer immediately below the units column should be established from the beginning.

VII. "Addition by endings," or "carrying the combinations to higher decades." In column addition the pupils must be able to add mentally a two-place to a one-place number. Counting by 2's, 3's, 4's, etc., beginning with 1, 2, etc., up to the number by which they are counting, is excellent preparation for this work. Much drill must be given on $\begin{array}{cccc} 2 & 12 & 22 & 32 \\ \underline{3} & \underline{3} & \underline{3} & \underline{3} \end{array}$, etc., $\begin{array}{cccc} 7 & 17 & 27 & 37 \\ \underline{4} & \underline{4} & \underline{4} & \underline{4} \end{array}$, etc.

Most of this drill should be oral and mental. Some schools prefer to extend the combinations to higher decades at the time they are first learned.

VIII. Columns involving "addition by endings."

$$\begin{array}{cc} 5 & 3 \\ 8 & 8 \\ 7 & 4 \\ 9 & 7 \\ & 8 \\ & 5 \end{array}$$

IX. Carrying. (*a*) Carrying one ten.

$$\begin{array}{cc} 26 & 24 \\ 38 & 23 \\ & 39 \end{array}$$

Some teachers prefer to start with cases such as $\begin{array}{r} 26 \\ 8 \end{array}$. Cases such as $\begin{array}{r} 26 \\ 38 \end{array}$ are, however, really easier and also more typical. Further, the pupils should be able to give the answer to $\begin{array}{r} 26 \\ 8 \end{array}$ by inspection and should not be permitted to form the habit of thinking 14, writing the 4 and carrying the 1.

(*b*) Carrying two or more tens.

$$\begin{array}{r} 37 \\ 88 \\ 19 \end{array}$$

X. Addition of three-place numbers (*a*) without carrying.

$$\begin{array}{r} 232 \\ 524 \end{array}$$

(*b*) with carrying from units to tens place.

$$\begin{array}{r} 358 \\ 217 \end{array}$$

(*c*) with carrying from tens to hundreds place only.

$$\begin{array}{r} 273 \\ 142 \\ \hline \end{array}$$

(*d*) with carrying in both places.

$$\begin{array}{r} 457 \\ 176 \\ \hline \end{array}$$

9
2
4
8
2
7
5
3
6
1
4
8
8

XI. Column addition with more than six addends. The new difficulty here arises from the increased attention span. Experiment shows that most pupils can keep their attention fixed on column addition for from six to eight additions. Then the attention wanders and the pupils if left to their own resources, often get completely lost. As a result pupils who can add columns of four or six addends successfully often fail utterly on longer columns. They must be taught to recognize this difficulty, to keep the proper place in the column with their pencil, and to remember the partial sum, while they let their attention wander momentarily and rest for an instant.

XII. Addition of addends having different number of digits. This difficulty is the same as met in adding two-place and one-place numbers. The habit of keeping the right hand column straight must be thoroughly established.

$$\begin{array}{r} 348 \\ 27 \\ 3849 \\ 9 \\ \hline \end{array}$$

Analysis of Difficulties in Subtraction

I. Combinations or facts.

$$\begin{array}{rr} 9 & 17 \\ 3 & 8 \\ \hline \end{array}, \text{ etc.}$$

II. Two-place numbers, subtrahend figures smaller than corresponding minuend.

$$\begin{array}{cc} 98 & 75 \\ \underline{23} & \underline{42} \end{array}$$

III. Zero difficulties.

$$\begin{array}{ccc} 56 & 28 & 30 \\ \underline{36} & \underline{10} & \underline{10} \end{array}$$

Some schools teach $\begin{array}{c} 30 \\ \underline{10} \end{array}$ or $\begin{array}{c} 3 \text{ tens} \\ \underline{1 \text{ ten}} \end{array}$ at the same time as $\begin{array}{c} 3 \\ \underline{1} \end{array}$ or $\begin{array}{c} 3 \text{ units} \\ \underline{1 \text{ unit}} \end{array}$.

IV. Two-place numbers, subtrahend figure larger than minuend in units place.

$$\begin{array}{cc} 53 & 40 \\ \underline{18} & \underline{23} \end{array}$$

V. Three-place numbers. (*a*) Subtrahend figures all less than the corresponding minuend.

$$\begin{array}{c} 375 \\ \underline{123} \end{array}$$

(*b*) Subtrahend figure in units place larger than corresponding minuend figure.

$$\begin{array}{c} 583 \\ \underline{256} \end{array}$$

(*c*) Subtrahend figure in tens place larger than the corresponding minuend figure.

$$\begin{array}{c} 427 \\ \underline{243} \end{array}$$

(*d*) Subtrahend figures larger than the corresponding minuend figures in both units and tens places.

$$\begin{array}{c} 623 \\ \underline{248} \end{array}$$

VI. Zero difficulty.

$$\begin{array}{r} 302 \\ 123 \\ \hline \end{array}$$

Problems of this type give difficulty particularly if the borrowing plan is used, in which the pupils think "Three from twelve is 9, two from nine is 7, one from two is 1." The new difficulty comes from the fact that there is a cipher in tens place in the minuend so there is nothing to borrow from, until ten is borrowed from hundreds place. This type of example will give less trouble if the pupils use the equal addition plan in which they think "Three from twelve is 9, three from ten is 7, two from three is 1."

VII. Subtrahend with fewer places than minuend.

$$\begin{array}{r} 375 \\ 92 \\ \hline \end{array}$$

VIII. Numbers with four or more places.

Analysis of Difficulties in Multiplication

I. Multiplication tables or facts.

$$\begin{array}{rr} 5 & 8 \\ 3 & 7 \\ \hline \end{array}$$

II. One-place multiplier and two-place multiplicand, no carrying.

$$\begin{array}{rr} 32 & 23 \\ 4 & 3 \\ \hline \end{array}$$

III. One-place multiplier and three-place multiplicand, no carrying.

$$\begin{array}{rr} 214 & 231 \\ 2 & 3 \\ \hline \end{array}$$

IV. Zeros in multiplicand.

$$\begin{array}{rrrrr} 40 & 30 & 220 & 204 & 400 \\ 2 & 3 & 4 & 2 & 2 \\ \hline \end{array}$$

Some schools teach $\frac{40}{2}$ or $\frac{4 \text{ tens}}{2}$ and $\frac{400}{2}$ or $\frac{4 \text{ hundreds}}{2}$ at the same time as $\frac{4}{2}$ or $\frac{4 \text{ units}}{2}$.

V. Carrying. (*a*) From units to tens.

$$\begin{array}{r} 125 \\ \times\ 3 \\ \hline \end{array}$$

(*b*) From tens to hundreds.

$$\begin{array}{r} 252 \\ \times\ 3 \\ \hline \end{array}$$

(*c*) In both units and tens places.

$$\begin{array}{r} 125 \\ \times\ 5 \\ \hline \end{array}$$

VI. Zero in product.

$$\begin{array}{r} 125 \\ \times\ 4 \\ \hline 500 \end{array}$$

When the result of a multiplication is 20 or 10, as in this example, pupils sometimes become confused and do not know what to put down, often putting down nothing or the number to be carried.

VII. Multiplication by 10.

VIII. Multiplication by 20, 30, 40, etc.

IX. Two-place multipliers. (*a*) Without carrying.

$$\begin{array}{r} 232 \\ \times\ 23 \\ \hline \end{array}$$

(*b*) With carrying.

$$\begin{array}{r} 96 \\ \times\ 72 \\ \hline \end{array}$$

X. Multiplication by 100.

XI. Multiplication by 200, 300, 400, etc.

XII. Three-place multipliers.

XIII. Zero in multiplier.

$$\begin{array}{rr} 756 & 342 \\ \underline{380} & \underline{308} \end{array}$$

Analysis of Difficulties in Division

I. Even division facts. $2\overline{)4}$, $3\overline{)12}$.

II. Short division, no carrying. $3\overline{)69}$, $3\overline{)129}$.

III. Uneven division facts. $3\overline{)17}$, $7\overline{)40}$.

IV. Short division, with carrying. (*a*) One to carry, no remainder. $3\overline{)48}$.

(*b*) Two or more to carry, no remainder. $6\overline{)84}$, $5\overline{)185}$.

(*c*) Carrying in two places, no remainder. $2\overline{)734}$.

(*d*) Remainder. $3\overline{)68}$, $2\overline{)357}$.

V. Zero difficulties. $4\overline{)840}$, $2\overline{)406}$, $4\overline{)816}$.

VI. Long division, two-place divisors, the first figure of the divisor being used as the trial divisor and giving the correct quotient figure. (*a*) Without carrying, in order to make the proper steps habitual.

$$\begin{array}{r} 32 \\ 21\overline{)672} \\ \underline{63} \\ 42 \\ \underline{42} \end{array}$$

(*b*) With carrying.

$$\begin{array}{r} 34 \\ 23\overline{)782} \\ \underline{69} \\ 92 \\ \underline{92} \end{array}$$

VII. Long division, two-place divisors, the trial divisor being one more than the first figure of the divisor and the trial quotient being the true quotient.

$$\begin{array}{r} 43 \\ 57\overline{)2461} \\ \underline{228} \\ 181 \\ \underline{171} \\ 10 \end{array}$$

If the second figure of the divisor is large (7, 8 or 9), the true quotient will be obtained in more cases by using one more than the first figure of the divisor as the trial divisor than when the first figure is used as it stands.

VIII. Long division, two-place divisors, the first figure of the divisor being used as the trial divisor, but the true quotient figure one less than the trial quotient.

$$\begin{array}{r} 7 \\ 43\overline{)2924} \\ 301 \end{array}$$

When the partial product is larger than the number from which it is to be subtracted, the pupils must learn to locate the difficulty, namely, that the quotient figure is too large. Much time will be saved if the pupils are encouraged, as soon as possible, to perform the multiplication mentally in order to see if the partial product is too large.

IX. Long division, two-place divisors, the trial divisor one greater than the first figure of the divisor and the true quotient one more than the trial quotient.

$$\begin{array}{r} 2 \\ 57\overline{)1767} \\ \underline{114} \\ 62 \end{array}$$

The pupils must be taught to compare the remainder with the divisor each time and when the remainder is greater

than the divisor they must know that the last quotient figure is too small and must be made one larger.

X. Long division, two-place divisors, quotient figure apparently one, but the partial product too large.

$$\begin{array}{r} 1 \\ 34\overline{)3230} \\ \underline{34} \end{array}$$

In such cases the pupils must learn to try nine and then eight if nine is too large.

XI. Zero in quotient.

$$\begin{array}{r} 305 \\ 27\overline{)8235} \\ \underline{81} \\ 135 \\ \underline{135} \end{array}$$

XII. Three-place divisors. Pupils must learn to use the first figure of the divisor as a trial divisor and to verify the quotient figure thus obtained by mentally multiplying the first *two* figures of the divisor by this quotient figure. The trial divisor should be one more than the first figure of the divisor if the second figure is 7 or larger, and in the verification the second figure of the divisor should be made one more if the third figure is 7, 8 or 9.

$$\begin{array}{r} 6 \\ 379\overline{)25586} \end{array}$$

Trial: 4 in 25 is 6. Verification: $6\times8=48$, $6\times3=18$ and 4 (carried from 48)$=22$.

The Development of Rules and Processes

Discovery of Tentative Rule or Process. In developing a new rule or a new step in some process, the teacher should select a simple typical case coming under the rule or process to be developed, and lead the pupils to think out the result, making use of the knowledge they already have.

If the course of study is properly organized, the pupils will always have sufficient knowledge to enable them to think out this example. Thus in working out the rule for multiplying decimals, the pupils can make use of their knowledge of multiplication of common fractions, or in developing the process of long division the pupils can use their knowledge of short division to get the results.

If the method used in getting the result in this one case is the same as the method that is to be finally used, or makes this method evident, then the next step is to reduce the work of getting the answer to its mere essentials. Thus, in developing the process of carrying in division, the teacher might start with the example $3\overline{)75}$. The pupils first think this out as "7 tens divided by 3 gives 2 tens and 1 ten over. One ten and 5 units make 15 units. Fifteen units divided by 3 gives 5 units." From this one example the pupils will usually discover what to do with the 1 that remains when they divide 7 by 3. If necessary, other similar examples may be taken to make the process clear. Then the pupils should reduce the process to its mere essentials and come to the tentative conclusion that all they need to think in solving the above example is "3 in 7, 2 and 1 over; 3 in 15, 5."

The method used in getting the result in the first example is not always the short "working method" but is often long and cumbersome. When this is the case, the pupils, having obtained the result by the long, thought process, should next try to find a short method of getting the correct answer. If this short method is not evident from the one example the teacher should take others and think them through with the pupils in the same way as the first. Then the pupils should examine and compare the several examples in order to find the short method. In developing the rule for multiplying decimals the teacher might start with the example $.2 \times .3$. This might be worked

out by the pupils by changing the decimals to common fractions. Thus $.3 \times .4 = \frac{3}{10} \times \frac{4}{10} = \frac{12}{100} = .12$. The method used in this case is too long, so the pupils try to find a shorter way of getting the answer. To aid them in this, the teacher takes other examples, such as $.2 \times .03$ and $.05 \times .07$, and the pupils work these as before by changing to common fractions. The teacher than writes the three examples and the answers on the board as follows:

$$.3 \times .4 = .12$$
$$.2 \times .03 = .006$$
$$.05 \times .07 = .0035$$

By comparing these the pupils discover that 12 is just 3×4, 6 is 2×3, and 35 is 5×7, and that the number of decimal places in the answer is equal to the number in the multiplier and multiplicand together, and formulate the tentative rule: "To multiply decimals, multiply as whole numbers and point off as many decimal places in the product as there are in the multiplier and multiplicand together."

Verification. The tentative method of procedure thus discovered, whether by thinking through one example or comparing several, should then be verified. This can be done by applying the short method to more examples and checking the results obtained, or by thinking through more examples in the same way as the first and also solving them by the short method to see if the results correspond. Thus the validity of the tentative process of carrying in division might be verified by taking other examples, such as $2\overline{)34}$, $4\overline{)92}$, etc., working them by the short process and checking the results obtained by multiplication. The tentative rule for multiplying decimals could be verified by working other examples in the same way as the first, namely, by changing to common fractions, and then working them by the short method to see if it gives the correct result.

Final Generalization. If the generalization is a rule, having once been made, it must be memorized for future use. It is better to have the pupils and teacher first agree on a form of statement and then have it memorized in that exact form. If the pupils are permitted to state the generalization each time in their own way it will lead to inaccurate statements and mistakes, besides wasting time. The exact form of the statement memorized is important, as it is to be used many times in the future and so must be stated in a convenient form. A rule is a set of working directions for performing a particular kind of task. As such, it must be brief, intelligible and specific. Thus, in stating the rule for multiplying a number by 12½ it is better to state it in the form, "To multiply a number by 12½ annex two ciphers and divide by 8," than in the form, "To multiply a number by 12½, first multiply it by 100 and then divide by 8." Further, the statement of the rule should always embody a description of the task to be accomplished, as well as directions for accomplishing it, in order to connect or associate in the pupils' minds the task on the one hand with the method of performing it on the other. Having worked out the rule for multiplying decimals it is not sufficient to formulate and memorize it in the form, "Point off as many decimal places in the product as in the multiplier and multiplicand together," as some text books do; but rather it should be stated and memorized in the form, "To multiply decimals multiply as whole numbers and point off in the product as many decimal places as there are in the multiplier and multiplicand together." Every rule should be stated and learned in the form, "To perform such and such a task, proceed as follows."

In developing a process the generalization is a brief summary of the essential steps to be taken in performing the process. Before leaving this phase of the develop-

ment and starting the application, the teacher should help the pupils to standardize the method of procedure and the words used in thinking through the process, and both of these should be reduced to an absolute minimum.

Application. The pupils should be practised according to the short, final method only and should never be practiced or drilled on the long thought forms. The sole purpose of thinking through a few examples is to discover the short method, and once discovered it should be used from that time on. To permit the pupils to use the long thought form after the short form has been discovered, is to establish a habit that is very difficult to break and which, if not broken, will be a serious handicap to the pupils' future success in arithmetic.

It is particularly important in developing rules and processes that the first applications be closely supervised by the teacher so as to prevent the pupils practicing on an incorrect method of procedure. A little care in the beginning may prevent mistakes which, if permitted to become habitual, will be very difficult to correct.

In the case of rules and processes it is usually necessary to make the first application to abstract drill examples rather than to applied problems. The pupils need to concentrate their whole attention on the carrying out of the rule or process and not have it distracted by the added difficulty of thinking out an applied problem. Thus, having formulated the rule for dividing by a fraction, the first application should be to abstract drill examples, such as $21 \div \frac{3}{4}$, $\frac{8}{9} \div \frac{2}{3}$, etc. The only thing the pupils have to think about here is how to apply the new rule. Having mastered the mechanics of the rule or process, it should then be used in concrete, applied problems in order to show the pupils its practical importance and to associate the rule with the kind of life situations in which it is to be used.

PART III

FIXING AND MECHANIZING, FACTS PRINCIPLES, RULES AND PROCESSES

CHAPTER I

METHODS OF FIXING—LAWS OF HABIT FORMATION

In the previous section the problem of how best to present new knowledge to the pupils has been discussed. In arithmetic this type of teaching occupies a comparatively small portion of the time devoted to the subject. The fact, principle, rule, or process once presented must be retained and made a part of the pupils' permanent mental equipment. It is not enough that the pupils understand and can use the process of long division the day it is first presented; they must in some way be made to retain this knowledge permanently so that they can use the process next year or any time in the future that the need for it may arise. Further, the process must be made automatic, mechanical; it is not enough that the pupils can perform the process of long division if they have time to think it through step by step, *they must be able to perform it mechanically without stopping to think.* Arithmetic has been called a habit-forming subject and habit formation is one of its most prominent and important, but at the same time not its only function. The fundamental facts, principles, rules and processes once presented must be memorized and mechanized. The type

of teaching by which this is done is one of the most prominent and important types that occurs in the teaching of arithmetic.

Two Methods of Fixing and Mechanizing

Fixing by Formal Repetition. After a new bit of arithmetical knowledge has been gained it may be fixed in mind in either of two ways, by *formal repetition* or by *use*. The first method, that of formal repetition, was formerly employed almost exclusively. The pupils were given the new knowledge, say a multiplication table, and then they were required to repeat it over and over again in a formal, abstract way until it was memorized and mechanized. Only then were they permitted to meet and use these facts in a concrete setting.

Fixing by Use. In recent years there has been a reaction against this mechanical, formal, abstract type of drill. It is usually attacked on the ground that it is uninteresting and deadening in its effects on the pupils, and that it is unnecessary and therefore wasteful of time. Those who take this point of view argue that as soon as the pupils acquire new information, such as a new set of facts, they should at once meet and use these facts in concrete situations or problems and that through frequent use they will eventually be memorized and mechanized. As an illustration of these two methods, suppose a class has just worked out a new set of multiplication facts. The teacher might then have them study and repeat the facts abstractly until memorized, and then use them in concrete problems. Or, as each fact is worked out, it might be written on the board (as $7\times8=56$) and then the pupils could work concrete problems involving these facts, getting the answers from the board. If the pupils work enough such problems the supposition is that the facts will ultimately be memorized and mechanized without any abstract repetition.

Comparison of the Two Methods. Each of these extreme methods has certain advantages and disadvantages. Formal repetition affords the maximum number of repetitions per pupil per minute and centers the attention on the abstract fact or process being repeated, and so it will undoubtedly fix and mechanize in the shortest possible time. There is the disadvantage, however, that the new fact or process is memorized abstractly and entirely unconnected with the concrete situation in which it is to be used later. The result, very often, is that the pupils know *how* to add but do not know *when* to add, that the pupils know how to divide but can not recognize a concrete division situation or problem.

On the other hand fixing by use alone has the disadvantage that it provides comparatively few repetitions per pupil per minute and centers the attention on the concrete situation rather than on the number relation or process being repeated, consequently it takes a long time to fix and mechanize anything by this method alone. Indeed, when one stops to consider the large number of facts, principles, rules and processes that must be mechanized, it becomes extremely doubtful if the pupils could possibly solve enough concrete problems to fix and mechanize all of them. It must be recognized, however, that fixing through use has the advantage that the facts, processes, etc., are learned and memorized in direct connection with the concrete situations in which they are used. As a result the pupils not only learn the fact or process, but at the same time form a mental connection between the fact or process and the corresponding concrete situation; the pupils not only learn how to subtract but at the same time unconsciously become familiar with the kind of problems that are to be solved by this process of subtraction.

Combination of Two Methods. Since success in life demands both an automatic control of the fundamentals and

the ability to use them in concrete situations, it follows that neither repetition nor use alone is sufficient. One extreme is as bad as the other. Much use is necessary *from the very beginning,* and this use not only develops the ability to apply arithmetic but also, to a larger extent than one is apt to think, serves to fix and mechanize the fundamentals. To insure this mechanization in the time at our disposal, use must be supplemented by a certain amount of abstract repetition. This repetition, although abstract, need not be monotonous and uninteresting in the hands of a skilful teacher. In fact, by the use of games, contests, time limit, standards, etc., it can be made intensely interesting. The term *drill* will be used in the rest of this book to mean the abstract repetition of a fact, rule or process in order to fix and mechanize, but repetition made interesting by the use of games, contests, etc.

Time Spent on Drill. A few ultra-modern schools have made the mistake of giving too little time to drill, but the majority of schools at the present time are probably spending too large a proportion of time on drill and too small a proportion on the applications of arithmetic. Teachers and superintendents lose sight of the fact that less abstract repetition will be needed if more concrete problem solving is done.

We have no experimental data to indicate what part of the time in each grade should be devoted to drill. Jessup and Coffman* obtained the opinions of 564 school superintendents on this question. The median† per cent of recitation time favored for strictly drill work by these superintendents was as follows:

Grade 1	43%	Grade 5	39%
" 2	50%	" 6	31%
" 3	52%	" 7	22%
" 4	45%	" 8	17%

*Jessup, W. A., and Coffman, L. D.—Supervision of Arithmetic.

†For definition of *median* see page 224.

Until we have positive experimental proof as to what constitutes the most advantageous distribution of time between drill and application, these figures probably afford the best available guide. The author is inclined to believe, however, that with proper methods of drill the time suggested for purely drill work by these superintendents, can be materially reduced.

Laws of Habit Formation

To say that the fundamental facts and processes of arithmetic must be fixed and mechanized is only another way of stating that the reactions involved in these facts and processes must be made habitual, that a set of specific habits must be formed. The laws of habit formation have been quite definitely established, and although different authors have stated them in different ways, in their essentials they are the same.

Repetition. It has already been stated that the chief method of habituating the facts and processes of arithmetic is *repetition* either in abstract form or in connection with concrete problems. Without repetition no habit is possible and it is the duty of the teacher to provide the pupils with enough opportunities for repeating the fundamental facts and processes of arithmetic to reduce them to habits.

Attention. A second factor in habit formation is *attention.* It is possible to form habits by repetition without attention, but it is a very long and wasteful process. The number of repetitions necessary to fix a habit can be greatly reduced if the repetitions are accompanied by a high degree of attention. The securing and keeping of the pupils' attention is one of the chief duties of the teacher in drill work. It is not repetition alone that counts, but *repetition with attention.*

Permit No Exceptions. In forming a habit it is very

important to permit no exceptions to occur. If one is trying to form the habit of getting up early in the morning and does so for five mornings but on the sixth remains in bed until 10 o'clock, the desired habit is no nearer formed than before he began. It is not safe to permit exceptions to a habit while it is being formed. In arithmetic, if the teacher wants to establish the habit of responding 8 to the situation 6+2, she must see to it that the pupils never respond to this situation by 7 or 9 or anything else except 8. If she wants to thoroughly establish the desired correct habit of procedure in "borrowing" in subtraction she must never permit the pupils to proceed in any other than the desired way.

Summary. The two most important duties of the teacher in drill work are: (1) To provide for repetition with attention; and (2) to prevent the occurrence of exceptions to the desired habits. These will be discussed in detail in the next two chapters.

CHAPTER II

METHODS OF SECURING AND KEEPING ATTENTION IN DRILL

The Attention Naturally Wanders. The mind is so constituted that the attention can be kept fixed on any given thing for only a comparatively short period of time. The natural tendency is for the attention to wander and fluctuate. At a given instant a thousand and one different sense stimuli, ideas, and thoughts are clamoring for possession of the pupils' minds and constantly tending to divert attention from the thoughts already there. The pupils can not prevent this and can not help being inattentive. In fact, there is no such thing as inattention; the mind is always focused on something, and what we call inattention is really attention to the wrong thing, to thoughts of going fishing instead of to $3\times4=12$, or to thoughts of a new dress instead of to $7\times4=28$. The teacher must expect the pupils' attention to fluctuate and must provide means of recalling it as soon as it wanders away.

The most important methods of securing and retaining the pupils' attention to drill work are: (1) The proper attitude and example of the teacher; (2) Proper motivation of the work; (3) Variation of procedure; (4) Use of a time limit; (5) Appeal to emulation, and (6) Short drill periods.

The Attitude and Example of the Teacher

Dr. Strayer* says, "The greatest single reason for lack of interest and attention on the part of the class is found

*Strayer, G. D.—A Brief Course in the Teaching Process.

in the indifference and lack of energy of the teacher." This statement is not too strong. All of the devices ever used or thought of will not succeed in arousing the pupils' interest unless the teacher herself is interested and shows it, and without the pupils' interest the teacher can not get a very high degree of attention. If a teacher wants her class to be interested, to pay attention, and to put all of their energy in the work, she must be interested herself and give the work all her attention and energy.

The Motivation of Drill

Definition. It has already been stated that abstract drill should be given only when needed. It is not enough, however, for the teacher to realize the need; she must make the pupils realize it also. Results obtained from drill depend not only upon repetition but also upon the degree of attention and the amount of energy given to the repetitions. If the pupils are convinced that they need drill on certain fundamentals they will give better attention and more energy to the repetitions, and as a consequence get the same results in less time or better results in the same time than they would if they simply drilled because they were told to drill. The teacher's first task in drill is to convince the pupils, if possible, that they need the drill. If the teacher drills only when she is convinced that drill is necessary, this should not usually be a very difficult task; it is simply a case of taking the pupils into her confidence and letting them see why she thinks they should drill.

Ways of Motivating Drill. There are several ways in which the need for drill may appear, and consequently several ways of motivating drill. The drill may be needed to enable the pupils to (*a*) solve applied problems, (*b*) master some fundamental process, (*c*) play some game, or (*d*) improve score, or reach the standard.

It is not sufficient to *tell* the pupils that they need drill

to solve problems or to play a game; they must be *shown*. The work must be so organized that the need for drill arises naturally from the pupils' daily work. Having obtained new knowledge, the pupils should immediately use it in problems and in games, and this attempt to use the fundamentals soon makes the need of practice on them evident to both the teacher and pupils.

Drill Needed to Solve Problems. As already stated, the fundamental, basic motive for all of the work in arithmetic is to be found in its practical applications. These practical applications, brought into the school as problems, form the connecting link between the arithmetic of the school room and the arithmetic of life and should constitute the backbone of the course. When new arithmetical knowledge is acquired it should immediately be used in problems, and whenever the pupils see that they are handicapped in the problem work by not knowing the fundamentals well enough, time should be taken from the applied work for abstract drill, returning to problem solving as soon as the drill has served its purpose.

In some cases the cycle from use through drill and back to use may be completed in a single recitation. More often, perhaps, it will cover several recitations; in solving problems one day the need for drill becomes evident, so the next day, or perhaps for several days, the time is devoted to drill, returning finally to the applied work. The teacher should never lose sight of the fact that the concrete applications of arithmetic are the reason for its being and consequently should be made the unifying element of the course, the drill coming only as it is needed to facilitate the application of the fundamentals to concrete situations. The problem motive should be emphasized from the first grade on. Thus, in the first grade, having discovered certain addition facts, the pupils "play store." Each pupil goes to the store and buys two articles, and

to find the total cost looks up the fact where it was written on the board when first discovered. The pupils are soon convinced that they need to memorize these facts because if they went to a real store the facts would not be written on the board. After the facts are partially memorized, if the pupils have trouble in finding the cost of two articles, they again see the need for more drill. In the eighth grade, high school, college or university, if the student gets into trouble in solving problems because he makes mistakes in division he should be helped to locate the difficulty and encouraged to overcome it by practice. The pupils are never too young or too old for the problem motive. It is the fundamental motive for all of the work in arithmetic.

Drill Needed on Details in Order to Master Some Process. The need for drill does not always arise *directly* from problem solving. For example, the pupils need drill on adding one-place numbers to two-place so that they can perform such additions mentally and mechanically without having to stop and think. The situation $\begin{array}{r} 27 \\ \underline{\ 8} \end{array}$ must bring the reaction 35 just as the situation $\begin{array}{r} 7 \\ \underline{8} \end{array}$ brings the reaction 15. This need does not grow directly out of problem work. If the situation 27+8 arose in a concrete problem the pupils would not need to know the result instantly, but would have plenty of time to think 8+7=15, 1+2=3. But in adding a column such as the one in the margin, the pupils must know instantly 6+7=13, 13+5=18, 18+9=27, and 27+8=35. The first addition involves only a knowledge of one of the primary addition facts, but in each of the others the pupils must know instantly the sum of a two-place and a one-place number. So the best way to motivate such drill is to show the pupils that this ability

8
9
5
7
6

is necessary to the mastery of the fundamental process of column addition.

Even here the ultimate motive is the problem, as drill on the mental addition of two-place and one-place numbers is necessary in order to master column addition, and a mastery of column addition is necessary in solving problems. The immediate need and consequently the most effective motive lies, however, in the process of column addition.

This type of motive is very useful in the intermediate and upper grades. If the pupils can not add, subtract, multiply or divide successfully it means that they have failed to master the facts or some one of the many abilities and habits necessary to the successful performance of these processes. The teacher must help them to locate the difficulty and encourage them to overcome it by practice.*

Drill Needed to Play Games. The game motive is particularly useful in the lower grades, although it should not be used to the exclusion of the problem motive. Even first grade pupils should know that they need to know 2+2 in real life as well as in games. In the intermediate and upper grades this motive can be used effectively in connection with the races and contests described in Chapter VI, Part III. If the pupils are interested in a game and find that they can not succeed at that game because they do not know the fundamental facts and processes well enough, they will willingly take the necessary drill. In the lower grades the game motive is really fundamentally the same as the problem motive, as the games are real life situations to the pupils and situations of the greatest and most compelling interest.

Drill Needed to Improve Score or Reach Standard. If the drill work is carried out in such a way that the pupils can keep their scores on a given kind of work from day

*For a list of the abilities and habits necessary to a mastery of the fundamental processes see Chapter 7, Part II.

to day they can be interested in trying to improve their scores. For example, third grade pupils in drilling on the addition combinations might be supplied with a sheet containing the combinations and be given one minute to write answers. At the end of the time the pupils correct their papers and record their scores. The teacher then tells them that they will try the same test again in a few days and that in the meantime they will drill on the combinations so they can make higher scores.

If the teacher can tell the pupils how much they should be able to do in the given time the motive is still more effective. Thus third grade pupils should, by the end of the year, be able to write the answers to 26 addition combinations in one minute. The setting up of a definite standard to be reached is one of the most effective motives with the average pupil. On the other hand, weak pupils who can never hope to reach the standard should not be discouraged by emphasizing the standard score too much. In their case the emphasis should be on improving their score, and this is an effective motive, as even the weakest pupil can improve. Standard tests are described and standard scores given in Chapter VII, Part III.

Variation in Method of Drilling

One of the strongest inborn tendencies in attention is to attend to the moving, the changing, rather than to the fixed and unchanging. The skilful teacher takes advantage of this inborn tendency to recall the pupils' attention when it starts to wander. She may not want to change the subject matter drilled upon, but she can change the method of drill. It should be remembered that the change of method is not made for the sake of the change, as long as a given method serves to hold the pupils' attention there is nothing gained by changing, but as soon as it fails to hold attention it should be dropped and another

method substituted. The wise teacher will never start a drill period without several drill devices prepared to use if needed. Fortunately there is no lack of variety in the methods of drill available. The work may be individual or group, written or oral, at blackboard or seat, and many variations and combinations of these types are possible. Further variety may be introduced by the use of games, contests, time limit, standard drill cards, etc.

Use of a Time Limit

One of the most effective methods of holding attention in drill is the imposing of a time limit. The drill may be either oral or written, but in either case the pupils are given a definite time and try to see how much they can do in that time. Limiting the time in drill has many advantages, one of the most important being that it forces the pupils to center their attention on the work in order to accomplish as much as possible in the given time. It thus provides brief periods of highly concentrated effort and attention, and that is what counts in drill.

Emulation—Games and Contests

Any drill device that appeals to the pupil's inborn desire to do better today than yesterday, to do as well as the other pupils in the class, or to have his class do as well as other classes, will be useful in holding attention and calling forth the pupil's best efforts. For a full discussion of the use of games in arithmetic, see Chapter VI, Part III.

Short, Snappy Drill Periods

The best of motives, the most enthusiastic and skilful of teachers, and the most interesting of drill devices will all fail to hold attention if the drill period is too long. Although the most effective length of drill period is not

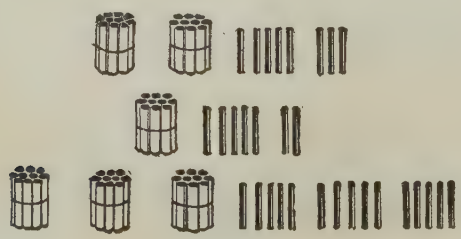
known, it has been shown* that a brief period produces better results than a long one. Probably drill periods of from one to ten or fifteen minutes are best, the exact time depending upon the grade, the subject matter, the class, the teacher, etc. A drill period is not too long as long as the teacher can keep up the attention, interest, and zeal of the pupils. When these are lost, further drill is a waste of effort at the best and may do positive harm.

*Kirby, T. J.—Practice in the Case of School Children.

CHAPTER III

HOW TO PREVENT THE OCCURRENCE OF EXCEPTIONS TO THE DESIRED HABITS

Be Sure the Pupils Understand the Exact Character of the Desired Habit Before Beginning Repetitions. Much depends on getting a correct start in drill work. The old saying "First impressions are the strongest" is psychologically sound. Before asking the pupils to repeat a fact or process it is essential that the teacher make clear to the pupils exactly what it is that they are to repeat, in order to insure the correctness of the first repetitions. In presenting the process of carrying in addition the first example, as $\begin{array}{r} 28 \\ 17 \\ \hline \end{array}$, might be worked out by the pupils by representing the two numbers on the top of the teacher's desk by means of sticks. Thus they would represent 28 by two bundles of ten each and eight single sticks and under it 17 by one bundle of ten sticks and seven single sticks. The seven sticks and the eight sticks make fifteen, which they would represent by fifteen sticks below the other two numbers. Then one ten and two tens make three tens, which they would represent by 3 bundles of ten under the other tens. They would then have on the desk the following:



The question would then arise as to what they might do with the 15 single sticks, namely, make a bundle of ten and put it with the other tens, giving the result

or 4 tens and 5 units, or 45. The teacher would then ask the pupils if they could add these numbers on the blackboard without using the sticks, and they would do so under her guidance thinking "7 units and 8 units make 15 units, but 15 units is 1 ten and 5 units, so we have 5 in units place. One ten and 1 ten are 2 tens and 2 more tens make 4 tens." The teacher next asks what they really need to think in adding the two numbers, bringing out that all that is necessary is to think "7 and 8 are 15 (writing the 5 and remembering the 1); 1 and 1 are 2; 2 and 2 are 4, (writing the 4)." Several examples should then be worked in this way, the teacher doing the work at the board until the pupils understand perfectly *the exact form of procedure that they are to repeat.*

Supervise All Drill Closely Until Habit Is Established. Even after the pupils understand what they are to repeat, mistakes are sure to occur which if not detected and corrected at once may persist and become so fixed as to make the establishment of the correct habit very difficult or even impossible. Until the habit is firmly established all drill must be done under the close supervision of the teacher and in such a way that mistakes can be detected as soon as they are made. Only then is it safe to let the pupils drill alone or by methods that can not be closely supervised. In the development of carrying in addition described above, after the teacher is sure the pupils understand exactly what they are to repeat, the first drill should be individual and oral so that the

teacher can tell just what the pupil is thinking as well as what he is writing. The author recently saw a recitation on the development of carrying in division. Under the guidance of the teacher the pupils thought out several examples such as $2\overline{)38}$ and $3\overline{)72}$, and at the end of the recitation seemed to understand the process, but had not determined just what they should think and write, and had had no opportunity to work examples for themselves. The teacher assigned a list of ten examples in the text to be worked for the next day, and then wondered why the pupils made so many mistakes. The author saw one paper on which the pupil had added the one to be carried to the next figure in all ten examples. The teacher's mistake here was two-fold. She not only started drilling before the pupils knew the form to be repeated, but in a way that made it impossible for her to detect mistakes until after they had been repeated over and over again. She not only had not started forming the correct habit but had permitted a wrong habit to start which had to be broken before the correct habit could be established. By having the pupils work individually at their seats or at the blackboard under her close supervision the teacher should have made sure that each and every pupil not only understood the process but could carry it out successfully and had worked enough examples to fix the process in mind temporarily, and then for several days the drill should have been entirely in class under the close supervision of the teacher. Then and then only would it have been safe to assign examples for independent drill.

Make Sure by Review and Study that the Pupils Have the Desired Facts or Process in Mind TODAY Before Starting Repetitions. Teachers too often forget that what the pupils knew and could do yesterday and what they know and can do today may be two entirely different things. At the time of the last drill recitation the pupils

may have had the facts or the process clearly in mind, but unless sufficient drill has already been given to establish the habit temporarily they may not have them in mind today. Nothing is gained by taking chances, the teacher should make sure that the pupils know the facts and that they understand and can perform the process *today* before asking them to drill. To do otherwise is to *invite* exceptions to the response that one is trying to habituate—to lay the basis for future trouble and mistakes. The author recently saw a recitation in which the teacher got into trouble by starting drill too soon. It was a drill recitation in the third grade on the multiplication table of 7's. This table had been developed and drilled upon the day before and at the end of the recitation the pupils knew the new facts as well as could be expected. Nothing had been done with these facts since the previous day. The teacher started the new recitation by calling on a pupil to repeat the table from memory. Several pupils were called on before the table was finally given correctly and in the meantime by actual count seven different mistakes had been made and one mistake had been repeated three times. Such errors are far more serious than we are apt to think, as they blaze the way, as it were, for future mistakes. This teacher had no one to blame but herself for this situation. The fact that the pupils could repeat the table at the end of the previous recitation is no evidence that they can do so today. It is far better to prevent incorrect responses than to try to correct them after they have been made. This teacher could have done this by having the pupils study the table for a few minutes at the beginning of the recitation, or before the recitation period began. Then she could have called for the facts from memory with reasonable certainty that they would be given correctly. No matter how careful a teacher may be the pupils will occasionally make incorrect responses,

but this is no excuse for teaching in a way that invites such errors.

Be Sure the Pupils Understand What They Are Doing. Although the pupils should always perform the fundamental processes mechanically, they should understand what they are doing. In the example $3\overline{)84}$ they should always think 3 in 8, 2 and 2 over; 3 in 24, 8; but at the same time they should understand what they are really doing when they "carry" the 2. The understanding of the process prevents many mistakes *while the habit is in its formative stage.* In the above example, if the pupil understands what he is really doing he is not apt to make the common mistake of adding the figure carried to the next place and thinking 3 into 8 goes 2 times and 2 over; 3 into 6 goes 2 times. Understanding of the process is one of the most valuable means of preventing the formation of incorrect habits of response in the fundamental processes. To teach a fact or process without letting the pupils see what they are really doing is just as bad as to let the pupil think it through every time. Both lead to future mistakes.

Be Sure Old Habits Are Well Established Before Introducing New. Many exceptions to the desired habits result from trying to go too fast, crowding one thing too closely on the heels of another, and introducing new facts or processes before the old ones are sufficiently mastered. As a result, the pupils become confused and make many mistakes. No better advice can be given to a teacher of arithmetic than "Make haste slowly."

Do Not Try to Form too Many Habits at One Time. The secret of success in habit formation is concentration on a few specific habits at a time. To try to habituate too much at one time leads to confusion, distraction of attention and dissipation of energy. In drilling on the fundamental facts only a few should be drilled upon at a time

until they are well established, then, when the object of further drill is simply to make the responses permanent and more rapid, it is safe to drill on a larger number of facts together. In forming habits in the fundamental processes attention and drill should be concentrated on a few points at a time and other difficulties left out until these are mastered and habituated. Thus the process of short division without carrying should be drilled on and mechanized before carrying is introduced.

Do Not Try for Speed Too Soon. While there is no conflict between accuracy and a reasonable degree of speed and one does not necessarily sacrifice accuracy to attain speed, yet the fact remains that if the teacher tries to speed up the pupils too soon accuracy is bound to suffer. Eventually a reasonable degree of speed is just as important as a high degree of accuracy, but *in drilling upon a new set of facts or on a new step in some process absolute accuracy is for the time not only the most important but the only end sought.* After the new habit is fairly well established the reaction can be gradually speeded up without loss of accuracy, but to strive for speed too soon or to try to increase the speed too rapidly is sure to result in a loss of accuracy.

Be Sure Work Is Not Too Difficult for Pupils. Very similar to the mistake of confusing the pupils by going too fast is that of failing to grade the work carefully. If the work is properly graded there should be a gradual increase in difficulty. Poor gradation often results in big jumps and the increase in difficulty is so sudden and great as to confuse the weaker or perhaps all of the pupils. Many mistakes result from giving work that is too difficult either for the class as a whole or for individual pupils. Not only should the work be carefully graded for the class as a whole, but it should also be suited in point of difficulty to the individual pupils. We no longer think that the only

fair way is to treat all alike, but rather that the only fair way is to treat each according to his capabilities and needs. Some pupils will need more drill on a given bit of subject matter than others and must get it in some way. To give all the same amount is to give the better pupils too much, and the poorer ones too little. The effect on one is just as bad as on the others. The better pupils can be excused from part of the drill and devote the time to problem solving, while the weaker pupils must be given extra drill. One of the greatest advantages of devices such as the Courtis and Studebaker Practice Tests described in Chapter VII, Part III, is that they provide for individual differences and give each pupil the amount of drill on a given topic that he needs.

Do Not Form Habits that Must be Broken Later. Do Not Attempt to Change a Habit Already Established Unless it is Positively Harmful. A habit once formed should never be changed unless it is failing to accomplish its purpose. If a pupil has formed the habit of subtracting by the take away method he should not be forced to change to the addition method. To attempt to break one habit of procedure and substitute another habit for it, is always to run the risk that neither habit will become fully established. The teacher is justified in trying to substitute a new method for an old, only when the old method is not working satisfactorily. This is evidence, usually, either that the habits involved have not been fully established, or that pernicious habits have been formed that render success with that method impossible. Then, often the best thing to do is to drop the old method and start over again at the beginning with the new, being careful to establish the proper habits and prevent harmful ones. Suppose in the seventh grade pupils are dividing decimals, using the method by which the decimal point in the quotient is placed over the point in the dividend. If a new pupil joins the

class and uses the method of counting the number of decimal places in the dividend and divisor and subtracting to get the number in the quotient, he should be permitted to continue using this method, *providing it works.* If he can not locate the decimal point in the quotient correctly, then the teacher would be justified in teaching him the other method.

If it is harmful to try to break old habits and establish new ones in their places it follows that we should avoid forming habits that will have to be broken later. Many teachers unconsciously permit pupils in the primary grades to form the habit of counting on their fingers instead of depending on their memory for the number facts. The pupils should discover some of the facts by counting, but after having discovered them the teacher must convince the pupils that it would be worth their while to *memorize* these facts and must insist that they give the facts from memory after that and not count them out each time. Perhaps it would be better never to permit the pupils to count on their fingers but to use other objects that the pupils will not always have with them. The fingers are the natural counting machine, but counting on them should be discouraged, as it is so easy for the pupils to form the habit of relying on them.

Teachers in developing a new rule or process sometimes have the pupils "think it out" for several days before reducing it to a habit. Thus, in long division, the pupils handle it as a thought process for several days and afterwards mechanize it. The pupils' practice on the fundamentals should be mechanical from the very beginning. This does not mean that new facts and processes should be given to the pupils arbitrarily. It is all right for the pupil to *discover* a new fact by counting or by thinking it out from some known fact, or to discover a new process, by thinking it out in detail, *but once arrived at, it should*

be used and drilled on mechanically. To permit a pupil to think out a fact or process each time he uses it is to run the risk of his forming the habit of thinking it out each time, and as a result he will not only waste time but will make more mistakes than if he does it mechanically. In a third grade drill recitation, recently observed by the author, the teacher called on a pupil for 6×8. He did not know, so the teacher asked $5\times8=?$ He remembered this and from it got that $6\times8=48$. The teacher was trying to teach the pupils to help themselves, but in so doing was running the risk of letting the pupils form the habit of thinking out the facts instead of depending on their memory for them. It would have been better if this teacher had called on another pupil to tell the first one what 6×8 is, had the first pupil repeat it, and returned to him frequently during the recitation to see whether he remembered the fact.

Another mistake frequently made is to teach one method first because it is easier for the pupil to understand (this usually means "easier for the teacher to teach") and later teach the more usable method. An instance of this occurs in the division of fractions. Some books first teach the pupils to divide fractions by changing to a common denominator as $\frac{2}{3}\div\frac{3}{4}=\frac{8}{12}\div\frac{9}{12}=8\div9=\frac{8}{9}$. After drilling on this method for awhile they teach the shorter method of inverting the divisor as $\frac{2}{3}\div\frac{3}{4}=\frac{2}{3}\times\frac{4}{3}=\frac{8}{9}$. The shortest and consequently the best method of dividing fractions is the method of inversion and consequently this is the method that should be taught in the schools and made habitual with the pupils. If the habit of inverting is to be fully established it must be formed from the very beginning. To form one habit and then try to break it and later form a better habit, almost invariably results in one of two things almost equally bad, either neither habit is fully established and as a result the pupils can not

successfully divide fractions, or else the earlier habit persists and the pupils go through life handicapped by using a long, roundabout method.

Avoid Use of Crutches. Pupils are often permitted to use auxiliary figures when a process is first introduced because the teacher thinks the use of such figures makes the process easier. Any device used temporarily while the pupils are weak, to make the work easier but later dropped, is a *crutch.* The trouble with such aids is that they are often not temporary but persist and become habitual. As all such aids take time they necessarily stand in the way of rapid work. Probably the safest course is never to permit the pupils to use such aids. It may make the process a little harder at first, but it is easier to get along without the crutch than it is to make the pupils stop using it when its use is no longer necessary. The most common examples of such crutches are (1) Writing down the figure to be carried in addition, multiplication and division, and (2) Crossing out and changing the figure in borrowing in subtraction. The author recently saw the following example in multiplication on the paper of a college freshman:

$$\begin{array}{r} 378 \\ \times\ 37 \\ \hline 2646 \\ 1134 \\ \hline 13986 \end{array}$$

Never Drill Without the Attention of the Pupils. In discussing attention the statement was made that to continue drill after the attention of the class is lost is not only a waste of effort but may do positive harm. Lack of attention will result in many mistakes being made and to continue the drill under these conditions will not only

do nothing towards establishing the desired habits but will tend to cancel the effect of the previous drill and may even start false habits that will make the formation of the correct habits very difficult. If for any reason whatsoever the teacher can not get or keep the pupils' attention she should stop the drill—to continue will do more harm than good.

CHAPTER IV

ACCURACY AND SPEED IN THE FUNDAMENTALS

Relation of Accuracy to Speed. Popular opinion is not at all agreed on the relation between accuracy and speed. One belief is embodied in the old saying, "Slow but sure." That is, many people believe that the two traits are more or less opposed to each other and that the slow calculator is apt to be more accurate than the fast, and vice versa. If this is true it would mean that we would have to sacrifice one trait for the other or else strike a happy medium in both. While many teachers hold this view and believe that an increase in speed is necessarily gained at the expense of accuracy, the majority of teachers probably hold the directly opposite view, *i.e.,* that the two traits necessarily go together, that the most rapid calculators are the most accurate; that any increase in accuracy brings about a corresponding increase in speed. If this were true, all that the teacher would need to do would be to strive in every way to increase accuracy and let speed take care of itself. There is the third possibility that the two traits are neither necessarily opposed to each other nor do they necessarily go together, but are independent and either may occur without the other, or both may occur together. The author has not found this belief nearly as common among teachers as the other two.

Fortunately, we have experimental evidence which proves conclusively which of these views is the correct one. Stone* showed that there is no necessary relation-

*Stone, C. W.—Arithmetical Abilities, Some Factors Determining Them.

ship between the two traits. In his investigation the school system that ranked first in accuracy was twelfth in rapidity, and the system that was second in accuracy was seventeenth in rapidity. On the other hand the system that was first in speed was also third in accuracy, and the system fourth in accuracy was also fifth in speed. These results show that while it is possible to get a high degree of speed without a corresponding high degree of accuracy and vice versa, it is also possible to get a high degree of both accuracy and speed. In short, accuracy and speed are independent qualities. This means that in teaching, it is not necessary to sacrifice accuracy to speed or speed to accuracy, as one can get both. It also means, however, that if the teacher is to get both she must try for both; speed does not come as a by-product of accuracy or accuracy as a by-product of speed.

Importance of Accuracy. A frequent criticism of our schools today is that the boys and girls coming from these schools are inaccurate; they can not add, subtract, multiply or divide with reasonable assurance of getting the correct result the first or even the second time they try. Anyone who has had any experience with school children in the grades or even in the high school must admit the truth of this criticism. Such a situation is a serious indictment of the arithmetic work being done in our schools today. Success in any work in life that demands any considerable use of arithmetic is impossible to one who is inaccurate in the fundamentals. *The teacher of arithmetic has no more important task than to do all in her power to develop a high degree of accuracy in the fundamental facts and processes.* Mere practice does not insure accuracy; constant and intelligent effort on the part of the teacher is absolutely necessary.

Importance of Speed. Popular opinion seems to regard speed as less important than accuracy. It is certainly true

that life demands a very high degree of accuracy and that one can never become too accurate, *i.e.*, more accurate than necessary to meet practical demands. It is also true that the situation is somewhat different with respect to speed. It is possible, by systematic practice, to gain a degree of speed that is useless in life simply because it is greater than is necessary to meet practical needs. It is equally true, however, that it is possible for one to be badly handicapped by the fact that his rate of speed in the fundamentals is less than life situations demand. In short, success in life demands at least a reasonable minimum of speed; to exceed this very greatly is unnecessary, but to fall very far below it is just as disastrous as to be inaccurate. Just what this minimum of speed is we do not know *exactly;* indeed, it may vary considerably in different vocations. We do, however, have certain standards of reasonable speed in the case of the four fundamental operations with integers. The Courtis Standard Research Tests, Series A and Series B, give standards of speed for the elementary facts and the complete processes respectively. While these standards are largely based on results actually obtained in the schools and not on what the world demands, they are much better than no standards at all. Indeed the speed given by the eighth grade standards is probably somewhat higher than required in practical work, so these standards at least set an upper limit on what constitutes reasonable and desirable speed. These Tests and other Standard Tests are described in detail in Chapter VII, Part III.

How to Develop Accuracy and Speed in the Fundamentals. In striving to develop accuracy and speed the teacher needs to know two things: (1) Neither accuracy nor speed is possible unless certain habits on which they depend are thoroughly established. (2) Accuracy and speed are themselves partly a matter of habit and the

teacher must go about forming these habits just as systematically as she does the habits involved in the fundamental facts and processes.

Habits Necessary to Accuracy and Speed

Among the more important habits that must be formed if the pupils are to attain reasonable accuracy and speed in the fundamentals are: (1) The habits involved in the fundamental facts and processes, (2) The habit of using a good form, (3) The habit of making legible figures, (4) Habits of brief thinking in the fundamental processes, (5) The habit of using short methods wherever applicable, (6) The habit of checking all calculations as soon as performed.

Fundamentals Must be Habituated. The most important habits conditioning accuracy and speed are the habits involved in the fundamental facts and processes. If these habits are not thoroughly established neither accuracy nor speed is possible. Many mistakes occur and much time is lost simply because the necessary habits are not fully formed, the necessary responses have not been made entirely habitual. Speed is impossible if the fundamentals have to be thought out each time. The roundabout thought methods sometimes taught as aids to memory are thoroughly pernicious. Some teachers, for example, teach their pupils in adding $\begin{array}{r} 8 \\ 9 \\ \hline \end{array}$ to add 10 and subtract 1 instead of adding 9. It takes no longer to remember the sum of 8 and 9 than it does to think 18 and every extra thought takes time.

Such devices inevitably slow up the work and are also apt to cause mistakes, as habitual responses are more accurate and rapid than those that depend on thought. The fundamental facts and processes must be thoroughly mechanized and habituated to insure speed and accuracy.

The formation of these habits has already been discussed in detail.

The remaining habits are formed in the same way as the other habits already discussed. The two most important things for the teacher to remember are: (1) That she must first convince the pupils of the desirability and necessity of the habit (provide a motive), and (2) That she must do all in her power to prevent any exceptions to the desired habit until it is fully established.

Correct Form. Many mistakes occur in the processes because of the incorrect or careless form used. The teacher must insist on an accurate and correct form in all written work from the very beginning and without exception, until the use of such a form becomes habitual. Perhaps the most important detail of correct form is the keeping of straight columns. This is not only important in addition but equally as important in subtraction, multiplication and division. The habit of putting figures in the proper column and *exactly* in the column must be thoroughly established.

Legible Figures. Another class of mistakes similar to the last arises from carelessness in forming the numerical figures or number symbols. Pupils write a 7 carelessly and later call it 1, or write 4 and call it 9, etc. Again it is the teacher's duty to insist that the pupils make legible figures from the beginning and to permit no exceptions until the habit of writing figures legibly is thoroughly established. Poor form and illegible figures also stand in the way of attaining reasonable speed. Even if no mistakes are made it is a waste of valuable time to have to stop and puzzle over a figure to determine whether it is a 2 or a 3, or whether it is in tens or hundreds column.

Besides permitting no exceptions the teacher must do all in her power to convince the pupils that proper form and legible figures *pay*. Every time a mistake occurs due to

either poor form or illegible figures, the attention of the whole class should be called to it, and if, as is often the case, there are pupils who can not hold their own in drilling with a time limit because they make illegible figures or are careless about form, or both, they and the class should be made to realize the cause of their being slow, and interested in trying to remove the handicap.

Brief Thinking in the Fundamental Processes. Many pupils are badly handicapped by having to think too many words in performing the fundamental processes. For example, in adding such as the one in the margin some pupils think "8 and 5 are 13, 13 and 9 are 22, 22 and 3 are 25," thinking each combination separately instead of thinking *results only,* namely, 13, 22, 25. In additive subtraction some pupils think "what added to 7 will give 9?" and then answer "2," instead of thinking "7 and 2." In practice the pupils should *see* the 9 and 7 but *consciously think* only the result "2." In division in an example such as $2\overline{)86}$ some teachers have the pupils think "How many 2's in 8?" and answer "4," or "2 goes into 8 four times" instead of the briefer form "2 in 8, 4." Again the pupils should see the 2 and 8, but consciously think only the result 4.

$$\begin{array}{r} 3 \\ 9 \\ 5 \\ \underline{8} \end{array}$$

In all four of the processes the teacher should see to it that the pupils form the habit of thinking results only and not the entire combination. From the very beginning she should standardize the terminology used in performing the fundamental processes and reduce it to an absolute minimum. Much of the early work on any new phase of a process must be oral so that the teacher can tell how the pupils are thinking and can make sure that they are forming the proper habits.

Again it is important that no exceptions be permitted and if the pupils have already formed the habit of thinking in too many words they must be convinced that they are

handicapped by this and interested in overcoming this handicap. One of the best ways of doing this is to give the class drill with a time limit. If a pupil is seriously handicapped he will soon discover the fact, as he will not be able to do as much as the other pupils in the given time.

The Habit of Using Short Methods. Many so called short methods should not be taught in the grades. Some of these are not really much shorter than the ordinary long method; others, although short when once mastered, apply to such special cases that the pupils will not have enough occasions to use them to justify taking the time necessary for their mastery. There are, however, some short methods that are so much shorter than the corresponding long methods and that can be used so frequently that every boy and girl should know them before leaving the grades.

The pupils should be shown these short methods, but to stop here is to accomplish nothing. The best of short methods is often not short until the pupils have practiced on it until it has become mechanical and they can use it as readily as the long method. Even then very often the pupils will not use the short method, simply because they have not formed the habit of using it. *Any short method that is worth teaching is worth drilling on until it is mechanized, and the pupils must be forced to use it in their everyday calculations until its use has become habitual.* The author recently saw a high school sophomore divide 3.1416 by 10 and use long division! Upon inquiry it developed that she knew the short method but had never formed the habit of using it. If the short method is really short the teacher can easily convince the pupils that the habit of using it is worth forming, and then it is simply a question of permitting no exceptions.

The Habit of Checking All Work. The habit of checking all work in the fundamentals is one of the most impor-

tant aids in attaining accuracy. Methods of checking should be taught early in the grades and the pupils should be required to use them until their use has become a habit. It should become second nature with the pupils to check all calculations as soon as performed; in fact, the calculation is not finished until they have determined that it is correct.

The pupils can easily be convinced of the importance of checking, both in school and in life. In school it makes them independent of the book and the teacher. By checking they are able to discover their mistakes before coming to class and so have the opportunity of correcting them. Checking makes answers in the back of the book unnecessary. Much might be said about the importance of checking calculations in life. The practical man who uses his mathematics in practical affairs must be able to rely on the accuracy of his results, and no matter how accurate he may be he will occasionally make mistakes, so in all practical work mathematical calculations must always be checked in some way.

Habits of Accurate and Rapid Work

Accuracy and speed, to a certain extent at least, are themselves habits and may be formed in the same way as other habits. Pupils are often slow and inaccurate for no other reason except that they have been permitted and have permitted themselves to fall into the habit of working slowly and carelessly. In trying to form in her pupils the habits of working accurately and as rapidly as possible, the two most important tasks of the teacher, again, are (1) To convince the pupils that accuracy and speed pay, that it will be worth their while to cultivate these habits; and (2) To permit no exceptions to the desired habits.

Convince the Pupils That Accuracy and Speed Pay. Not only do many pupils get through the grades without

being able to add a column of figures with any assurance of getting the correct result, but in many cases they are not at all concerned about their inability to do so. Indeed many pupils in the upper grades seem to feel that accuracy in computation is of no importance and that it is below their dignity to concern themselves with such elementary things. Such an attitude is the direct result of the kind of teaching these pupils have had. *If the teacher expects accuracy in computation from her pupils she must do everything in her power to impress them with its importance both in and out of school.*

As previously stated, popular opinion regards speed as less important than accuracy. The teacher must overcome this attitude and convince the pupils that reasonable speed and accuracy are both essential to success in school and in life. Among the more important methods of doing this are the following: (1) Attitude of teacher, (2) Example of teacher, (3) Games, (4) Standards, and (5) Show the importance of accuracy and speed in life.

Attitude of Teacher. Many teachers apparently do not themselves realize the importance of accuracy, and seem fairly well satisfied with inaccurate work. Some teachers even do things that positively encourage inaccuracy. Among these may be mentioned the practice of some teachers in grading drill examples. An example such as the one in the margin, having four places in the answer is sometimes graded 75 per cent correct if three of the four figures are correct, 50 per cent if two of the four are right, etc.

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 3785
 4862
 8647
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Still more common is the practice of allowing practically full credit in the solution of applied problems for the correct method of solution even if the result is wrong. Such practices as these inevitably result in the pupils unconsciously coming to believe that accuracy in computation is not very important. The result obtained in adding

the numbers in the example given above is either *right* or *wrong;* it can not be 50 per cent right. A mistake in one of the four figures makes the whole answer wrong. In working an applied problem the result is the only thing that counts, *socially speaking,* and the pupils must be made to realize this. If our problem is to find the number of acres in a tract of land the accuracy of our result is what counts; the process by which that result was obtained is of no importance. If in buying a tract of land the surveyor made a mistake in figuring the number of acres and one paid for more land than he got, the fact that the surveyor had used the correct *method* in computing the acreage would be poor consolation.

We can not expect the pupils to attach very much importance to accuracy of computation if the teacher shows that she does not, by giving a high grade on any example or problem in which a mistake has been made in calculation. If the pupils are to be made to realize the importance of accuracy the teacher must demand absolute, 100 per cent accuracy *all of the time,* and must be satisfied with nothing else. If the teacher does not realize the importance of speed the pupils are not apt to do so. The teacher should constantly demand reasonable speed as well as absolute accuracy and should not be satisfied, or permit the pupil to be satisfied, with work that is accurate but unreasonably slow. The author once saw a drill recitation in a sixth grade in which the pupils were having a review drill on the multiplication facts. The teacher held up before the class a pack of cards, each card having two numbers printed upon it. The first pupil to recite made several mistakes and sat down realizing that he had made a poor showing. The next pupil went through the entire pack of cards without making a single mistake, but was very slow and had to stop an appreciable instant to think before giving each answer. The teacher praised this pupil, and

she sat down beaming and evidently thought that she had done all that could be expected of a sixth grade pupil. Such teaching is bound to result disastrously as far as speed is concerned.

Example of the Teacher. An inaccurate teacher will never make accurate pupils. In the first place, such a teacher may fail to detect mistakes made by the pupils More serious than this, however, is the unconscious effect of the teacher's example. The teacher may do all in her power to impress the importance of accuracy on the pupils, but her efforts will constantly be thwarted, as the pupils have before them daily a person whom they probably regard as fairly successful in life who has attained to that success in spite of inaccuracy. Such a teacher should overcome her disability or, if that is impossible, should get into some work where it will handicap no one but herself.

The teacher must be able to work rapidly herself and set the pupils an example. If the teacher is slow the pupils will also be slow. Not only does the example of the teacher have its bad effect but the speed of much of the drill work is limited by the speed of the teacher.

Checks. One of the best ways to convince pupils that accuracy pays is to teach methods of checking calculations and to require the pupils to use them until their use has become habitual. Checks not only enable the pupils to detect and correct their own mistakes but also automatically punish inaccuracy. If the pupils are drilling on column addition, instead of assigning ten examples to be worked without checking, the teacher could give the class five columns to be added and checked. If no mistakes are made the pupils have to add ten columns (re-adding each a second time as a check), but every time a mistake is made the pupils are punished by having to add a column a third time in order to detect the mistake. If required to check and correct all calculations, the pupils soon learn that

it is easier in the long run to be more careful and "get it right" the first time. They soon find that accuracy pays.

Games. No game should be used in arithmetic that does not demand absolute accuracy. The pupils soon find that they can not win or even make a creditable showing in these games if they are inaccurate. Success in many of the games depends upon being able to work rapidly as well as accurately, this being particularly true of number races of all sorts. Such games do much to convince pupils that it pays to be able to work both accurately and rapidly. For a discussion of the use of games, see Chapter VI, Part III.

Standards. Nothing will do more to increase the speed and accuracy of the pupils than to set up definite standdards. Pupils often do not realize how slow and inaccurate they are until they are tested by standard tests and given the opportunity to compare their scores with those of other schools and with the standard scores. Drill devices such as the Studebaker Economy Practice Tests and the Courtis Standard Practice Tests, and standard tests of all kinds keep a definite standard constantly before the pupils, and as a result do much to develop speed and accuracy. For a discussion of these tests see Chapter VII, Part III.

Importance of Accuracy and Speed in Life. Unfortunately, pupils sometimes get the impression that success in life does not depend upon the same qualities as success in school. In every possible way the teacher must drive home the importance of accuracy and speed in the fundamentals to success in life. This can be best done, perhaps, in connection with the applied problems in arithmetic. The problem should provide the connecting link between the school and life; indeed, it should be an actual life situation, or the description of such a situation, brought into the school-room. In dealing with such a problem

the pupils and teacher should consider not only its arithmetical relations but also its social setting and significance. The pupils must understand the social situation that gave rise to the problem and must put themselves in the place of the person who would have to solve such a problem. In solving a problem about making change the pupil should imagine himself a clerk in a store who actually has to make change and if he makes a mistake he must consider the possible results, such as having the amount of the mistake deducted from his wages, possible loss of his position, etc. In solving problems arising in the construction of a house, let the pupils put themselves in the place of the contractor and bid on the job of excavating the cellar. A mistake in calculation may wipe out the supposed profit and substitute a loss.

It does little or no good for the teacher to *tell* the pupils that they will be handicapped in life by inaccuracy and slowness; they must be *shown* the results in concrete cases brought within their own experience.

Permit No Exceptions. In order to establish a habit of accurate work the teacher must demand absolute, 100 per cent accuracy all of the time and must not let her sense of justice or anything else make her deviate from this course. Practices such as those previously mentioned of giving credit for the process in the solution of concrete problems when the answer is wrong, and giving part credit when part of the figures of the answer are wrong, are disastrous. No exceptions must be permitted *until the habit of accuracy is established.* Then if the teacher's sense of justice leads her to give a pupil credit for thinking through a difficult applied problem correctly, although his answer is wrong because of some slight mistake in his calculations, no harm will be done; but until the habit of accurate work is established the teacher should *permit no exceptions whatsoever.*

Avoid Methods of Drill that Demand Less than the Pupil's Maximum Rate of Speed. Time Limit. Many teachers make the mistake of using methods of drill that develop the habit of working at less than the maximum rate of speed. The teacher must be sure that the methods of drill she is using are suited to her grade and to her pupils and that they force the pupils to work at their maximum rate of speed.

Most of the drill after the facts or processes are fairly well established must be done with a time limit. That is, the pupils should be given a certain definite period of time to see how much they can do. Whenever possible, a definite standard should be set up for the pupils to strive to reach. At first this standard should be comparatively low, but it should be gradually increased as the pupils' speed increases. It is only by forcing the pupils to always work at their maximum possible rate of speed and gradually increasing this rate that we can establish the habit of speedy work. To permit exceptions by using methods of drill that do not demand the pupils' utmost speed will render the task of forming the habit extremely difficult if not impossible.

CHAPTER V

MISCELLANEOUS POINTS ON DRILL

Organization of Facts Drilled Upon

The number of individual facts to be drilled upon in arithmetic is quite large, there being one hundred in addition alone and as many in each of the other processes. In drilling upon a series of facts, such as the addition combinations, the teacher should keep several things in mind.

Include All the Facts. In drilling upon the primary facts in the four fundamental processes it is very important that none be omitted. It would be perfectly possible to drill day after day on addition facts and miss some entirely. The work must be systematized and organized in such a way as to make this impossible.

Emphasize the More Difficult Facts. These facts are all equally important but not all equally difficult for the pupils to memorize. For example, 5×7 is usually much more easily remembered than 8×7. It is not always possible for the teacher to predict which facts are the most difficult, indeed it will vary somewhat with different classes and pupils. The teacher should encourage the pupils to find out which facts are most difficult for them and pick out these facts for extra drill.

Drill on the Facts in All Forms that May be Met. The primary facts in the four operations may be met by the pupils in a variety of forms. Two forms, often called the horizontal and vertical, are quite common. In the following the horizontal form is given first and is followed by the corresponding vertical form.

$9+3=12$, $7-2=5$, $3\times7=21$, $12\div3=4$,

$$\begin{array}{r} 9 \\ +3 \\ \hline \end{array} \qquad \begin{array}{r} 7 \\ -2 \\ \hline \end{array} \qquad \begin{array}{r} 7 \\ \times 3 \\ \hline 21 \end{array} \qquad 3\overline{)12}$$

In life the pupils may meet both of these forms and so should meet both in school.

Emphasize Form that Occurs Most Frequently. Although the pupils may meet both forms, they are not of equal importance. Only enough drill should be given on $9+3=12$, $7-2=5$, $3\times7=21$ and $12\div3=4$ to enable the pupils to understand these forms when they meet them. The pupils should receive most of their drill on the facts in the form in which they will meet these facts in the processes, and that is $\begin{array}{r} 9 \\ +3, \\ \hline 12 \end{array}$ $\begin{array}{r} 7 \\ -2, \\ \hline 5 \end{array}$ $\begin{array}{r} 7 \\ \times 3 \\ \hline 21 \end{array}$ and $3\overline{)12}$.

Vary the Order of the Facts in Drilling. There has been much discussion as to the best order of teaching the fundamental facts. This question was considered in Chapter VI, Part II. No matter what order is used in presenting the facts for the first time, in drilling upon them the order must be constantly changed. Formerly the multiplication facts were learned in the form of tables and each table was learned as a whole. The result was that the pupils knew the table as a whole instead of knowing each fact individually, and as a result if they wanted to know 7×8 they would sometimes have to start at the beginning of the table and "run it down." If $7\times8=56$ is to be learned as a separate fact, 7×8 and 56 must always be presented together, and they must be presented in varying surroundings. If the facts $6\times8=48$ and $7\times8=56$ are always kept together there is danger that the connection will be made between 56 and 48 instead of between 56 and 7×8, so that the pupils will have to go back to $6\times8=48$ in order to get $7\times8=56$.

Distribution

The results obtained from drill depend not only upon the number of repetitions but also upon the distribution of the repetitions. Dr. Ebbinghaus in his experimental investigation of memory (1885) showed that "with any considerable number of repetitions a suitable distribution of them over a space of time is decidedly more advantageous than the massing of them at a single time." More recent investigations have confirmed this conclusion. The purpose of the drill work in arithmetic is to fix certain facts, rules and processes and make them permanent possessions of the mind. Suppose one wants to memorize a poem *so that he will never forget it.* He studies it until he can repeat it perfectly at the time, but after an interval of time, long or short, as the case may be, he finds that he can no longer repeat it. Apparently he has forgotten the poem completely, but in reality he has not; it has simply slipped the threshold of his memory and when he memorizes it again he finds that it takes fewer repetitions. After learning it the second time he forgets it again, *but the rate of forgetting is less rapid.* Thus, by learning and forgetting, then after an interval relearning and forgetting, after another interval relearning and forgetting again, etc., the rate of forgetting eventually becomes so slow that for all practical purposes the poem has become a permanent possession of his mind.

If we are to insure permanent retention of the fundamental facts and processes of arithmetic we must provide for drill on them at recurring intervals throughout the school course. For this reason the fundamentals should be introduced as early as possible and drilled upon throughout the grades. Indeed, many high schools are finding it advantageous to provide systematic drill in the fundamentals of arithmetic in the first two years. It is not

known exactly what length of drill period is most advantageous or exactly what length interval should elapse between successive drill periods. Kirby* found that a brief period produces better results than a longer period. It has also been found that the intervals between drill periods should be gradually increased and the length of the drill periods themselves gradually decreased.

The author once heard a third grade teacher declare that she was going to give her pupils so much drill on the multiplication tables that they would never forget them. This teacher wasted much time in trying to realize this ambition and then the pupils needed more drill in the fourth grade. One can not establish these habits all at once and should not expect to. Teachers frequently complain that the pupils they get from the preceding grade have forgotten certain things called for in the course of study. This should not be surprising, indeed it is the regular and natural order of things. The teacher should expect to have to teach over much that has been taught before, but at the same time should remember that the fact that it has to be taught over is not necessarily any reflection on the previous teacher, nor does it mean that the time spent on the topic the year before was wasted. The teacher has to bring it above the threshold of memory again, but that will take much less time than it would to teach it if it were entirely new, and besides things usually have to be taught several times if they are to be retained. The author recalls that he first learned the process of extracting square root in the eighth grade, used it a few days and promptly forgot it. The process was learned again in algebra in the freshman year of high school, forgotten, learned again in plane geometry, forgotten, learned again in solid geometry and physics, for-

*Kirby, T. J.—Practice in the Case of School Children.

gotten and learned again as a freshman in college and retained until this day, although practically never used.

One of the excellent features of the Courtis and Studebaker Practice Tests described in Chapter VII, Part III, is the provision they make for recurring drill with shortened periods.

CHAPTER VI

GAMES

Types of Games. Perhaps one of the most striking innovations in the teaching of arithmetic in recent years has been the introduction of games. Like everything else that is new, their use has been overdone and they have not always been wisely chosen or used. A knowledge of the various types of games and of the purposes for which they may be used to advantage is necessary for their intelligent choice and use. There are four types of games in common use: (1) Games proper, (2) Imaginative or "make-believe" games, (3) Races and contests, and (4) Mathematical recreations, curiosities and puzzles.

Games Proper

This type includes all parlor and outdoor games that involve numbers. They are games such as the children play outside of school and not merely drill in an interesting way. The following are examples:

1	8	4
5	9	6
3	7	2

Bean Bag. There are many forms of this game. In one, the accompanying diagram is drawn on the floor with chalk. Several feet away a line is drawn, behind which the pupils stand and throw a bean bag at the center square marked 9. Each pupil in turn is given two throws, and

his score is the sum of the two numbers hit. In order to get this score the pupil must state the sum correctly. Either individual or team scores may be kept.

"I Am Thinking of Two Numbers." The teacher says "I am thinking of two numbers whose sum is 7. What are they?" A pupil guesses, "Are you thinking of 6 and 1 are 7?" and the teacher replies, "No, I am not thinking of 6 and 1 are 7," etc., until some pupil guesses the combination correctly, when he takes the teacher's place and proposes a problem.

Pussy Wants a Corner. The pupils are arranged in a large ring with one in the center. The pupils in the ring are given numbers, each number being given to two different pupils. The one in the center calls "2 and 3" and the two pupils having the number 5 try to change places, the one in the center trying to get one of the vacant places while they are exchanging.

The well-known games of hop-scotch, dominoes, ring-toss, lotto, parchesi, tenpins, and the spinning game in which balls are knocked into numbered holes by a spinning top all belong to this type. Such games are particularly suitable for use in the lower grades and may be played in school, at recess, or taught to the pupils and played outside of school hours.

Imaginative or "Make-Believe" Games

In this type the drill itself is the game, but it is put into an attractive form by the pupils imagining or pretending that they are doing something else. The added interest arising from rivalry may or may not enter into such games. The following games belong to this type:

Climb the Ladder. The combinations to be drilled upon are placed on the rungs of a ladder drawn on the blackboard. As they give the combinations the pupils play they are climbing, first up the ladder and then down again.

Stepping Stones. The teacher draws on the board the picture of a river with a row of stepping stones from one bank to the other. On each stone is written a combination. The pupils play they are crossing the river as they give the combinations.

Other games of a similar character are "Walking the Ties to Boston," "Nimble Squirrel Climbing a Tree," "Fish Pond," etc. Games of this type can be made intensely interesting to pupils in the lower grades, as they appeal to the child's instinctive love of pretending. In order to be interesting, however, the teacher and pupils must enter into the imagined situation with their whole souls and must think and talk in terms of this situation. Giving addition combinations written on a ladder or on stones is no more interesting than giving the same combinations written in a simple list on the blackboard unless one really pretends that he is climbing a ladder or crossing a stream. As an illustration, suppose a ladder leaning against an apple tree is drawn on the blackboard. On the tree are apples, and on each apple a combination. The pupils pretend they are climbing the ladder to get an apple. If a pupil hesitates too long in giving a combination on the ladder he is "stuck" and another pupil must give him a "boost." If he gives a combination incorrectly he "falls off the ladder." When he gets to the top rung of the ladder he is to pick an apple by giving the combination. If he chooses an easy combination it is a small apple, while a difficult combination is a large one. Then he has to climb down again, which is often harder than going up. In the stepping-stone game the pupils may imagine they are crossing a stream. If they make a mistake they slip and fall into the water and someone must volunteer to rescue them by going out over the stones and giving the combinations correctly.

Races and Contests

Three Types of Emulation. In this type of game the drill itself is again the game, but is made interesting by being put into the form of a race or contest. The interest arises wholly from the appeal made to the spirit of emulation or rivalry. The rivalry may be (*a*) between individuals, (*b*) between groups or teams, or (*c*) with self.

Rivalry Between Individuals. There are several objections to the use of rivalry between individuals. One type of race involving this kind of rivalry is as follows. The teacher places on the board two sets of examples of equal length and difficulty. Two pupils go to the board to race and at a given signal start solving the examples. The first one through is the winner, providing all of his examples are correct. Such a race is open to the objection that it is very wasteful of time, as only two pupils out of the class are getting the drill, while the others are merely watching. To correct this difficulty all of the pupils might be sent to the board; or, if board space is too limited to permit of this, each pupil might be given a sheet or card containing examples to be worked at their seats. In this way all of the pupils are participating in the race, but the rivalry is still individual, as each pupil is trying to finish first. As a result the slower pupils are forced to compete with the faster, which is bad for both, as the slow ones become discouraged and the others lose interest. *Any method of procedure that forces or encourages the comparison of pupils of unequal ability is fundamentally bad.* Further, rivalry between individuals tends to foster habits and ideals of individualism and selfishness, and often leads to ill feeling among the pupils. Individual rivalry, if used at all, must be used sparingly and with a full knowledge of its weaknesses and possible dangers.

Group Rivalry. Rivalry between groups or teams, if properly handled, possesses none of the weaknesses of indi-

vidual rivalry. Enough teams can be formed so as to include every member of the class so all the pupils can participate in the contest, and by carefully selecting the teams they can be made of approximately equal ability, the slow pupils being balanced by quicker ones. In this way, as each team has a chance to win, there is an incentive for each pupil to do his best. Such rivalry fosters the social spirit among the pupils, encourages them to work for the general good of a group rather than for their own individual good. Further, it brings to the teacher's aid a very powerful force—social pressure—that can be used very effectively in dealing with the lazy and indifferent pupil.

A superintendent recently told the author the following incident: Three fourth grade boys were brought before him for fighting. The case looked rather serious, as apparently two of the boys had attacked the third. Upon inquiry the following explanation was obtained from the boys: The fourth grade teacher had her arithmetic class divided into two permanent teams, the "Reds" and the "Blues." Score was kept on all drill work and at the end of each week the team having the highest score was declared the winner. The three boys were all members of the "Reds" and their team had been losing consistently for several weeks, due to the fact that one of the boys was not trying. To use the boys' own words, what they had done to this boy was only a sample of what they would do if he didn't "get busy." The superintendent decided not to interfere, but kept in touch with the situation through the boys' teacher, who reported that the lazy boy immediately "got busy," with the result that the "Reds" won their full share of the weekly contests.

Self Emulation. Self emulation has in the past been somewhat neglected in our schools, but in recent years its use has become more general. The following is an illus-

tration of this kind of work. The pupils of a sixth grade have been drilling on long division, and as a test of their ability their teacher prepares a list of examples of approximately equal difficulty and gives the pupils a certain time, say ten minutes, to see how many of the examples they can solve correctly. The number of examples given is more than any of the pupils can solve in the given time. At the end of the ten minutes the papers are corrected and each pupil makes a record of the number attempted and the number correct, which constitutes his score. This score is kept and later, having drilled on long division in the meantime, a similar list of examples is given and each pupil tries to improve his previous score by solving more correctly in the same time.

This kind of work is particularly good in that it emphasizes self-improvement, which is one of the most useful motives and ideals that the school can inculcate. The pupils can not all hope to "beat the other fellow," perhaps at his own game, and if they try they are foredoomed to failure. Most of the pupils must learn to be satisfied and content with doing their best, but it should be a constantly growing best. Self emulation is just as effective with the slow pupils as it is with the quicker ones; there is always the possibility of improvement, no matter how slow the pupil may be.

This type of rivalry, if skillfully used, can be made just as interesting to the pupils as either of the others. The author recently saw a drill lesson of this type in a fourth grade. At the end, each pupil reported his score for that day together with his best previous score. The girl who had made the poorest score was the happiest; she had five examples correct, whereas before she had never been able to get more than three. By emphasizing self-improvement this girl was spurred on to redoubled efforts, whereas she would have been very bady discouraged if

she had been led or permitted to compare her work with that of the other members of the class.

Two Kinds of Races. Aside from the kind of rivalry involved, races may be of two kinds: (*a*) All of the contestants do the same amount of work and the winner is judged according to the time taken; (*b*) all of the contestants work for the same length of time and the winner is judged by the amount done. The first type of race has been most common in arithmetic, probably due to the fact that it is the type with which we are most familiar in running, horse, bicycle and automobile racing. The second type of race is not unknown even here, however, as witness the one-hour walking, the six-day bicycle, and the one-hour automobile races.

Comparison of Two Types. The second type of race possesses many advantages over the first for school use. In the first place it is more economical of time. If the pupils are all given the same amount of work, as in the first type, the quicker pupils will get done first and have nothing to do, while they wait for the slowest pupil to finish. On the other hand, in the second type of race all of the pupils are busy for the entire period, all start and stop together at a given signal.

In the second place, it is much easier to determine the relative standing of the pupils in the race with a time limit. The pupil having the largest number of examples *correct* at the end of the given time is first, the one having the second largest number *correct* is second, etc. In order to emphasize the importance of absolute accuracy some teachers give a score, say of five, for each correct example, and penalize by deducting one from the total score for each one that is incorrect. Thus a pupil who worked eight examples in the given time and had six correct would get a score of thirty less a penalty of two for inaccuracy, giving a net score of twenty-eight.

Finally, this type of race lends itself more readily to keeping scores by teams. If the first type is used and there are five members on each side it is difficult to determine which team is the winner if the members finish in the following order:

Team A—1 3 5 8 10
Team B—2 4 6 7 9

Sometimes the "order numbers" are added for the two teams and the team having the smaller sum is called the winner. In this case Team A would win, having a sum of 27 to B's 28. Or else, the winner is declared to be the team all of whose members finish first. Under these conditions Team B would be the winner, as the last member of this team finished before the last member of Team A. Neither of these methods of determining team scores is entirely satisfactory, and the second is positively harmful. It throws the entire burden of losing on the shoulders of the slowest pupil. Imagine how anxious each team would be to have this slowest pupil and imagine the effect on such a pupil. Other difficulties arise when some of the examples are solved incorrectly. These are not insurmountable but are, to say the least, very troublesome.

None of these difficulties occur in determining team scores in the second type of race. The team scores are simply the sums of the scores made by the individual members of the team, *only those examples being counted that are solved correctly.* Further, the two teams need not be equal in number, as the average team score can easily be found. Suppose two teams, one of four and the other of five members, have scores of examples correct as follows:

Team A—7 8 5 3
Team B—4 5 9 6 2

Team A is the winner with an average of 5¾ as opposed to 5⅕ for Team B. This gives the pupils excellent prac-

tice in figuring averages and shows the necessity for obtaining such averages. The use of averages is so important in life that everything possible should be done to emphasize them in school.

Mathematical Puzzles, Curiosities and Recreations

Mathematical puzzles, curiosities and recreations of all kinds have been very much neglected in our schools, perhaps because they are suitable for the intermediate and upper grades, and the teachers in these grades are just commencing to realize that they can use games of the proper sort to just as great advantage as the teacher in the primary grades. The following will serve to illustrate this type of game or recreational material.

Guessing Game. This is a well known game and can be played in many different ways. One simple form is as follows: Pupil A tells pupil B to think of some number and remember it, add 8, add the original number, divide by 2, subtract the original number and the answer will always be 4. For example, start with 9, adding 8 gives 17, adding the original number (9) gives 26, dividing by 2 gives 13 and subtracting the original number (9) gives 4. The operations used and the answer obtained may be varied, but the operations must always be so chosen as to eliminate the original number. This becomes clear if we represent the operations algebraically. Let n stand for the number, then the results in the successive steps are n, $n+8$, $2n+8$, $n+4$, 4.

Mysterious Addition. Have some one write a number and under it a second number having the same number of digits. Then, write a third number such that the sum of its digits and the corresponding digits of the number above will always be 9. Have some one else write a fourth number and then write a fifth in the same way as before. The

sum of the five numbers will always be the first number written with the digit 2 written in front of the other digits and with two subtracted from the last digit. The numbers in heavy black and the answer are written by the person performing the trick, the others by anyone else. The explanation is simple. The second and third numbers taken together make 9999 and the fourth and fifth make the same. The sum of 9999 and 9999 is 19,998 to be added to the original number. But 19,998 is just 2 less than 20,000, so adding 19,998 is the same as adding 20,000 and subtracting 2. This is done readily by placing a 2 in front of the other digits and subtracting 2 from the last digit. The numbers used may have any number of digits but no one of them can have any more than the first number written. Instead of five numbers, seven or nine may be added in the same way, in which case 3 and 4, respectively, must be prefixed and subtracted instead of 2.

3789
2495
7504
8726
1273
23787

Russian Multiplication. The following method of multiplication, said to be in use among the Russian peasants, requires only the ability to add and to multiply and divide by 2. To multiply 85 by 94 write the two numbers at the head of two columns. Each number in the first column is divided by 2; if the quotient is not an integer only the integral part is written. This is continued until the quotient is 1. In the second column each number is multipled by 2 until as many numbers are obtained as in the first column. Then the numbers in the second column that stand opposite *odd* numbers in the first colmun are added, which gives the required product. In this case the numbers in heavy black type are added, giving 7990 as the product of 85 by 94.

85	**94**
42	188
21	**376**
10	752
5	**1504**
2	3008
1	**6016**
	7990

The explanation is simple and depends on the principle that if in the product of two numbers one of the factors is multiplied by any number and the other factor divided by the same number the product remains the same. The following should make the explanation clear:

$85\times94=84\times94+94=42\times188+94=21\times376+94=20\times376+376+94=10\times752+376+94=5\times1504+376+94=4\times1504+1504+376+94=2\times3008+1504+376+94=1\times6016+1504+376+94=6016+1504+376+94.$

Puzzle Problems. Puzzle problems such as the following and the two given in the first chapter of this book rightfully belong to this class of games.

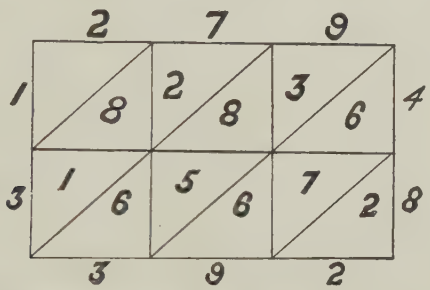
1. Three travelers met at an inn, and two of them brought their provisions along with them; but the third, not having provided any, proposed to the other two that they should all eat together, and he would pay them for his proportion. This being agreed to, A produced 5 loaves, and B 3 loaves, which the travelers ate together and C paid eight equal pieces of money as the value of his share, with which the other two were satisfied but quarreled about the division of them. Upon this, the affair was referred to an umpire, who decided the dispute justly. What was his decision?

2. A hare starts 50 leaps before a greyhound, and takes 4 leaps to the hound's 3; but 2 of the hound's leaps are equal to 3 of the hare's. How many leaps must the hound make to overtake the hare?

Used occasionally as *puzzles* such problems are a valuable means of increasing the pupils' interest in the subject; used constantly under the pretense of being practical or life problems they cannot be condemned too strongly.

Old Methods of Calculating. In the upper grades pupils are interested in abacus reckoning and old methods of calculating with the Hindu-Arabic numerals. The follow-

ing method of multiplication, at one time very common in Europe, will serve as an illustration:



The example shown is $48 \times 279 = 13392$.

Purposes for Which Games Are Used

Games are ordinarily used to serve one or more of four purposes, (1) as drill, (2) as a motive for drill, (3) to encourage accuracy and speed, or (4) to create an interest in mathematics in general and arithmetic in particular.

Drill Games. A game that is used for drill must be above all things economical of time; it must provide for a maximum number of repetitions per pupil per minute. Besides this, it must be interesting or it will fail of its purpose, namely, to lend variety and interest to the drill work. Games of the second and third type are usually suitable for drill work, as the game is really nothing but the drill put into an interesting form.

Motivating Games. On the other hand games of the first type are apt to be too slow and wasteful of time to be of much use for drill. They may, however, serve another purpose, namely, provide a reason or motive for drill to be done in some other way. The first requisite of a game to use for this purpose is interest. Even if it is somewhat wasteful of time and even if the pupils get little or no drill from the game itself, if the game is very interesting and the pupils like to play it the teacher is justified in

taking the time occasionally. The pupils will gladly drill for a week in order to play a favorite game.

Games used for drill may be used frequently in order to lend variety and interest, but games used solely as a motive for drill should only be used occasionally. Games of the first type are usually too wasteful of time to use except as a motive for drill work. The pupils get very little drill from Bean Bag, Tenpins, etc., but if they enjoy these games an occasional period devoted to them is time well spent, as it will provide a reason for drilling for at least a week before.

Races, skillfully planned, usually give enough repetitions per pupil per minute to be used for drill, but one of the most interesting races, namely, the relay race, is an exception.

Relay Race. The pupils are divided into two teams. Drill examples of equal difficulty are written on separate slips of paper or cards and placed face down on the teacher's desk in two piles, one for each team. At a given signal the first member of each team takes the top card and, going to the board, solves the example it contains and returns to his seat. As soon as he is seated, the second member of the team takes the second card and goes to the board, etc. Before solving his own example each contestant has the privilege of correcting mistakes made by any previous member of his team. The first team to finish all of its examples *correctly* is the winner. If the team that finishes first does not have all the examples correctly solved the second team is declared winner, providing they have made no mistakes. If both teams have made uncorrected mistakes it is no race. Such a race is intensely interesting, and if planned several weeks ahead gives life and spirit to drill that might otherwise be deadly dull.

The holding of an old-fashioned "Ciphering Match" is another excellent method of motivating drill. Such a

match can take many forms, of which the following is only one.

"Ciphering Match." The contest may be on just one kind of "ciphering," as addition, or it may involve everything the pupils have had, including applied problems as well as abstract examples. The teacher, or whoever conducts the match, should have a list of problems and examples on separate cards. The first member of Team A is given a problem and goes to the board and solves it. If he solves it correctly in a given time, say two minutes, he takes his seat and another problem is given to the first member of Team B; if not, the same problem is given to the representative of Team B, and so on until it is solved correctly in the given time. Whenever a team member fails to solve his problem he is "ciphered out" and withdraws from the contest. Problems are given alternately to the members of the two teams until one team is entirely "Ciphered Out."

Relay races and "Ciphering Matches" between different rooms or different schools or between two chosen teams from the same room are both interesting and profitable. If the contest is to be between different rooms or schools preliminary contests or try-outs should be held in each room to choose its representatives.

Accuracy and Speed. In choosing a game for a given purpose one other thing must be considered. No game should ever be used unless it demands absolute accuracy, but some games also put a premium on the speed of the response while others demand accuracy only. Each is good if used in the proper place, but bad if used in the wrong place. When a pupil is first learning a set of facts such as the multiplication facts or mastering a new process, as carrying in addition, it is disastrous to try to hurry him too much, as it will lead to "guessing" and establish incorrect connections and habits that will stand in the way of

the mastery of the facts or process and lead to mistakes and inaccuracies in the future. Games such as "I am thinking of two numbers," "Bean Bag," "Climb the Ladder," etc., can be used at this time, but games that demand speedy responses, such as "Pussy Wants a Corner," or a race of any kind, are entirely out of place.

On the other hand, when the facts or process have once been fairly mastered, the games used must demand quickness of response. To use a game at this stage that gives the pupil all the time he wants to stop and *think* before he makes a response tends not only to develop habits of slow response but also to prevent the mechanization of the fact or process by giving the pupil time to think it out.

Using Games to Create an Interest in Mathematics. All of the different types of games help to create an interest in arithmetic and mathematics, but mathematical puzzles, curiosities and recreations of all kinds are particularly good for this purpose. These recreations do not usually afford much drill themselves or demand drill in order that they may be enjoyed, but they are very valuable and should not be neglected. Many a pupil has had his first interest in mathematics awakened and stimulated by these recreations and through them has discovered a hitherto unsuspected interest in and aptitude for the subject. Among the author's most vivid recollections of his early education are the Friday afternoons that his eighth grade teacher sometimes devoted to puzzle problems. Many of these recreations are suitable for the intermediate grades and many more for the upper grades or the junior high school.

No attempt has been made in this chapter to give an extensive list of games. The purpose has been rather to make clear the ways in which games can be used to advantage and the different types of games that are suitable for various purposes. For the benefit of teachers who are

interested in making a collection of games a bibliography follows:

Bibliography of Games

Number Primers.

Harris, A. V., and Waldo, L. M.—First Journeys in Numberland; Scott, Foresman and Co., Chicago.

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CHAPTER VII

MEASURING THE MASTERY OF THE FUNDAMENTAL FACTS AND PROCESSES GAINED THROUGH DRILL

The Problem Is Complex. The measurement of the results obtained through drill on the fundamental facts and processes of arithmetic is a rather complex problem. In 1908 Dr. C. W. Stone* showed that there is no such thing as "ability in arithmetic." Proficiency in the subject depends not on any simple ability but rather upon a number of specific abilities. His conclusions have been corroborated by a number of more recent investigations.

Even a mastery of one of the fundamental processes with integers involves many different abilities. In Chapter VI, Part II, the four fundamental processes were analyzed, and according to the analyses given there the process of addition involves twelve different abilities and habits, subtraction eight, multiplication thirteen, and division eleven; making in all forty-four abilities and habits necessary to a mastery of the fundamental processes with integers alone. Successful work with fractions and compound numbers demands many more, so it is evident that the problem of measuring abilities in the fundamental processes of arithmetic is a complex one.

Purpose of Standard Tests. In general, Standard Tests are used in two ways: (*a*) By superintendents and supervisors at stated intervals (usually at the beginning, middle and end of the year) as a means of judging the efficiency of the teaching of arithmetic in a system as a whole

*Stone, C. W.—Arithmetical Abilities and Some Factors Determining Them.

and by the individual teachers. (*b*) By the class-room teacher, at frequent intervals throughout the year, as a means of setting a definite standard before the pupils and of locating individual and class weaknesses. In this book, which is intended for the teacher, only the second of these two uses is considered and the Standard Tests are treated throughout as a teaching device to be used by the class-room teacher.

The teacher of arithmetic needs to know two things about her pupils' "abilities" in the fundamentals: (1) What types of examples they can handle successfully, and (2) how well they can handle them; that is, how accurately and how rapidly. The teacher needs to know both of these things if she is to make her teaching effective. All of the following standard tests are designed to enable the teacher to find out one or both of these things.

Courtis Standard Research Tests, Series A*

Description. The Courtis Standard Research Tests, Series A, consist of eight tests. Only five of these will be considered here. Tests Nos. 1, 2, 3, and 4 are merely tests of the pupils' knowledge of the fundamental facts in the four operations of addition, subtraction, multiplication, and division, respectively. On each of these tests the pupils are given one minute to write answers to as many of the combinations as they can. Test No. 7 is intended as a test of the pupils' abilities in the four fundamental processes with integers. The time for this test is twelve minutes. The tests follow.

*Reprinted by permission of the author.

"Measure the efficiency of the entire school, not the individual ability of the few"

SCORE

No. attempted.....

No. right.....

ARITHMETIC—Test No. 1. Speed Test—Addition

Name..... *School*..... *Grade*.....

Write on this paper, in the space between the lines, the answers to as many of these addition examples as possible in the time allowed.

1 7 9 3 2 3 7 6 0 4	8 9 7 8 2 1 9 6 0 5	1 6 9 0 4 2 6 5 1 2	1 3 6 0 3 5 8 9 7 2
3 4 7 0 3 1 6 9 8 5	5 8 6 9 4 1 3 5 0 3	1 4 8 0 2 6 7 9 5 7	1 2 5 6 7 4 9 8 0 2
1 3 8 2 3 7 9 5 0 7	1 8 6 0 5 9 4 7 2 4	2 9 7 4 5 2 3 8 0 2	4 8 9 5 3 1 8 7 0 6
9 2 5 0 6 1 8 7 4 3	3 7 9 0 4 3 4 8 6 5	7 4 8 0 3 1 9 6 0 4	2 4 5 1 6 1 8 9 0 2
5 9 6 7 5 5 2 8 0 3	6 9 8 1 2 1 4 7 1 3	4 8 5 0 7 4 2 6 9 3	1 6 7 0 2 8 4 5 3 6
8 9 7 8 5 1 9 6 0 2	1 4 9 0 4 6 7 5 1 2	1 7 9 3 2 3 7 6 0 4	1 2 6 0 3 6 7 9 7 2

"Measure the efficiency of the entire school, not the individual ability of the few"

SCORE
No. attempted.....
No. right.....

ARITHMETIC—Test No. 2. Speed Test—Subtraction

Name..... *School*..... *Grade*.....

Write on this paper, in the space between the lines, the answers to as many of these subtraction examples as possible in the time allowed.

8 11 12 5 10 0 9 7 1 2	9 7 11 8 12 9 3 6 1 3	6 11 15 10 12 0 7 8 9 4	1 9 13 4 12 0 7 8 3 6
4 10 13 10 9 4 7 5 1 4	2 7 13 3 10 1 5 7 2 5	5 8 17 6 11 5 6 9 4 8	1 6 15 4 8 1 3 9 2 3
7 8 16 9 11 0 5 7 1 4	4 9 16 7 11 0 6 9 1 3	9 7 14 5 11 0 2 8 4 2	5 12 15 5 16 0 9 6 3 8
2 10 11 9 14 0 6 5 8 7	2 12 15 3 10 2 8 7 1 3	6 7 17 6 9 6 4 8 1 2	8 13 12 8 13 8 9 5 7 4
7 6 13 10 9 7 2 6 8 3	0 5 14 7 8 0 2 5 6 2	3 8 14 6 10 0 4 9 5 4	3 9 14 4 18 3 5 6 1 9
6 11 15 10 12 0 7 8 9 4	8 11 12 5 10 0 9 7 1 2	1 9 13 4 12 0 7 8 3 6	9 7 11 8 12 9 3 6 1 3

"Measure the efficiency of the entire school, not the individual ability of the few"

SCORE
No. attempted.....
No. right.....

ARITHMETIC—Test No. 3. Speed Test—Multiplication

Name..... *School*..... *Grade*.....

Write on this paper, in the space between the lines, the answers to as many of these multiplition examples as possible in the time allowed.

4 2 7 4 9 1 9 6 0 5	2 3 9 0 7 1 3 6 5 4	3 4 9 0 5 2 7 8 2 6	9 5 4 7 6 1 2 8 0 5
1 2 7 0 8 6 8 7 6 3	8 2 7 5 4 1 6 9 0 6	1 2 8 1 5 9 5 7 1 3	2 5 6 0 7 3 5 9 8 3
1 3 7 6 5 2 5 8 0 9	6 2 8 9 5 1 7 4 0 7	3 2 6 0 8 1 4 7 1 5	4 3 9 8 6 2 6 7 0 4
1 4 8 0 4 5 4 9 3 5	1 3 6 0 3 7 4 8 0 9	7 3 9 2 4 1 8 9 0 3	1 6 8 0 9 4 2 8 7 3
2 5 4 3 7 2 8 9 0 5	5 8 6 0 5 1 2 3 9 4	1 6 9 1 7 8 2 4 0 2	1 9 8 0 3 3 2 6 4 7
3 4 9 0 5 2 7 8 2 6	9 5 4 7 6 1 2 8 0 5	4 2 7 4 9 1 9 6 0 5	2 3 9 0 7 1 3 6 5 4

"Measure the efficiency of the entire school, not the individual ability of the few"

SCORE
No. attempted.....
No. right.....

ARITHMETIC—Test No. 4. Speed Test—Division

Name..... *School*..... *Grade*.....

Write on this paper, in the space between the lines, the answers to as many of these division examples as possible in the time allowed.

9)9 3)21 6)48 1)1 5)10	3)9 4)32 6)36 2)0 7)28	1)8 5)30 8)72 1)0 9)36	2)6 4)24 7)63 6)0 8)32
1)4 5)35 9)45 2)2 3)12	8)8 4)28 5)40 2)2 8)16	5)5 4)36 9)54 8)0 4)12	1)5 2)16 8)48 1)2 9)27
3)6 4)20 7)49 1)3 2)8	1)7 2)10 7)42 1)1 6)18	5)0 3)24 9)63 2)4 8)24	6)6 3)27 8)64 1)2 4)16
7)7 2)18 6)42 3)0 7)21	5)5 2)14 8)40 9)0 5)15	4)4 3)15 9)81 7)0 6)12	4)4 6)30 8)56 1)0 7)14
1)5 3)18 9)72 4)0 6)24	1)4 2)12 5)45 3)3 4)8	1)6 7)35 6)54 1)3 5)20	1)9 5)25 7)56 3)3 9)18
1)8 5)30 8)72 1)6 9)36	9)9 3)21 6)48 1)1 5)10	3)9 4)32 6)36 2)0 7)28	2)6 4)24 7)35 6)0 8)32

"Measure the efficiency of the entire school, not the individual ability of the few."

SCORE
No. attempted.....
No. right.....

ARITHMETIC—Test No. 7. Fundamentals

Name..... *School*..... *Grade*.....

In the blank space below, work as many of these examples as possible in the time allowed. Work them in order as numbered, writing each answer in the "answer" column before commencing a new example. Do no work on any other paper.

No.	Operation	Example	Answer	Rt	
1	Addition	*a* 32+130+725=..(Write answer in this column)			
		b 152+8001+120+3023=.....			
2	Subtraction	*a* 4748−136=.....			
		b 362974−221801=.....			
3	Multiplication	2201×231=.....			
4	Division	375024÷312=.....			
5	Addition	8225+134+2900+5004+4050 +363=.....			
6	Subtraction	62132104−38396767=.....			
7	Multiplication	56804×564=.....			
8					
9	Division	15826992÷4=.....			
10	Division	3333220÷436=.....			
11					
12	Addition	78558+68696+59393+73859+			
13		66773+86696+68887+98951=			
14	Subtraction	16535424−8875657=.....			
15	Multiplication	89576×876=.....			
16					
17	Division	51495423÷7=.....			
18	Division	5361384÷679=.....			
19					

Limitations. By themselves the first four tests measure only *one* of the many abilities involved in the mastery of the fundamental processes. A pupil may know his facts perfectly and still not be able to solve more complex examples, he may know his addition facts and not be able to add columns, he may know his multiplication facts and not be able to multiply 384 by 6, etc. Because of these limitations Test No. 7 and later the four tests of Series B were devised to test ability to perform the complete processes. The four tests of Series B, described below, are more satisfactory for this purpose than Test No. 7 and should be used whenever time permits.

Courtis Standard Research Tests, Series B*

Description. There are four tests in this series, one each on addition, subtraction, multiplication and division. Since these tests are intended to measure the pupils' accuracy and speed in the processes with integers, all of the examples in a given test are of the same type and of equal difficulty. Thus the number attempted in the given time gives the measure of speed and the number correct divided by the number attempted gives the per cent of accuracy. The tests follow.

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"Measure the efficiency of the entire school, not the individual ability of the few."

SCORE
No. attempted.....
No. right.....

Arithmetic. Test No. 1. Addition

You will be given eight minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

927	297	136	486	384	176	277	837
379	925	340	765	477	783	445	882
756	473	988	524	881	697	682	959
837	983	386	140	266	200	594	603
924	315	353	812	679	366	481	118
110	661	904	466	241	851	778	781
854	794	547	355	796	535	849	756
965	177	192	834	850	323	157	222
344	124	439	567	733	229	953	525
537	664	634	572	226	351	428	862
695	278	168	253	880	788	975	159
471	345	717	948	663	705	450	383
913	921	142	529	819	174	194	451
564	787	449	936	779	426	666	938
932	646	453	223	123	649	742	433
559	433	924	358	338	755	295	599
106	464	659	676	996	140	187	172
228	449	432	122	303	246	281	152
677	223	186	275	432	634	547	588
464	878	478	521	876	327	197	256
234	682	927	854	571	327	685	719
718	399	516	939	917	394	678	524
838	904	923	582	749	807	456	969
293	353	553	566	495	169	393	761
423	419	216	936	250	491	525	113
955	756	669	472	833	885	240	449
519	314	409	264	318	403	152	122

Name..... *Grade*.....

"Measure the efficiency of the entire school, not the individual ability of the few."

SCORE
No. attempted.....
No. right.....

Arithmetic. Test No. 2. Subtraction

You will be given four minutes to find the answers to as many of these subtraction examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy but it is more important to have your answers right than to try a great many examples.

107795491	75088824	91500053	87939983
77197029	57406394	19901563	72207316
160620971	51274387	117359208	47222970
80361837	25842708	36955523	17504943
115364741	67298125	92057352	113380936
80195261	29346861	42689037	42556840
64547329	121961783	109514632	125778972
48813139	90492726	81268615	30393060
92971900	104339409	60472960	119811864
62207032	74835938	50196521	34379846
137769153	144694835	123822790	80836465
70176835	74199225	40568814	49178036

Name..... *Grade*.....

"Measure the efficiency of the entire school, not the individual ability of the few."

SCORE
No. attempted.....
No. right.....

Arithmetic. Test No. 3. Multiplication

You will be given six minutes to work as many of these multiplication examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

8246	3597	5739	2648	9537
29	73	85	46	92
4268	7593	6428	8563	2947
37	640	58	207	63
5368	4792	7942	3586	9742
95	84	72	36	59
6385	8736	5942	6837	4952
48	502	39	680	47
3876	9245	7368	2594	6495
93	86	74	25	19

Name..... *Grade*.....

"Measure the efficiency of the entire school, not the individual ability of the few."

SCORE
No. attempted.....
No. right.....

Arithmetic. Test No. 4. Division

You will be given eight minutes to work as many of these division examples as possible. You are not expected to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

$25\overline{)6775}$	$94\overline{)85352}$	$37\overline{)9990}$	$86\overline{)80066}$
$73\overline{)58765}$	$49\overline{)31409}$	$68\overline{)43520}$	$52\overline{)44252}$
$37\overline{)14467}$	$86\overline{)60372}$	$94\overline{)67774}$	$25\overline{)9750}$
$68\overline{)39508}$	$49\overline{)28420}$	$52\overline{)21112}$	$73\overline{)33653}$
$28\overline{)23548}$	$54\overline{)48708}$	$39\overline{)32760}$	$67\overline{)61707}$
$45\overline{)33795}$	$76\overline{)57000}$	$93\overline{)28458}$	$82\overline{)29602}$

Name..... *Grade*.....

Limitations. (*a*) These tests deal with integers only and give no measure of ability with common and decimal fractions and with compound numbers. (*b*) They do not give a complete measure of the four processes with integers. They determine whether or not a pupil or class is up to the standard in solving examples involving all of the different difficulties occurring in the completed process but if the pupil or class is not up to standard these tests do not locate the difficulty. If the pupil has not all of the abilities necessary to the successful performance of the complete process of addition of integers these tests will not determine which he has and which he has not. Even when supplemented with Series A these tests are incomplete. A pupil might be up to standard on Test 1, Series A (Addition combinations), and fail utterly on Test 1, Series B, because knowledge of the combinations is only one of the many abilities involved in column addition.

The Woody Arithmetic Scales†

Description. These scales were recently devised to indicate "the type of problems and the difficulty of the problems that a class can solve correctly."* There are two series of four scales each; Series B is made up of examples taken from Series A but contains only half as many examples. These scales are fundamentally different from the Courtis Tests. Since they are intended to measure the type and difficulty of the examples that a class can solve correctly, the examples, instead of being of equal difficulty, start with the easiest possible example and gradually increase in difficulty to the last, and instead of all being of the same type they are of different types, including integers, common and decimal fractions and compound numbers. Twenty minutes is allowed for each of the tests

†Published by Teachers College, Columbia University. Reprinted by permission of author and publisher. *See page 210.

in Series A. This time is so long that "Most of the children will have finished before that time. Those who do not have done, in all probability, all they can; at least they have taken as much time as it takes the average class to complete the test."* Thus the element of speed does not enter. The score of an individual pupil is a statement of the particular examples which he has solved correctly. The score of a class is the degree of difficulty of an example in the scale that is solved correctly by just fifty per cent of the class. The relative degree of difficulty of the examples has been determined and a value assigned to each. The tests of Series A follow.

Series A—Addition Scale

By Clifford Woody

Name .....
When is your next birthday? .....
How old will you be? ..... Are you a boy or girl? .....
In what grade are you? .....

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
2	2	17	53	72	60	3+1=	2+5+1=
3	4	2	45	26	37		
	3						

(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
20	21	32	43	23	25+42=	100	9
10	33	59	1	25		33	24
2	35	17	2	16		45	12
30			13			201	15
25						46	19

*Woody, Clifford—Measurements of Some Achievements in Arithmetic.

(17)
199
194
295
156

(18)
2563
1387
4954
2065

(19)
$ .75
1.25
.49

(20)
$12.50
16.75
15.75

(21)
$8.00
5.75
2.33
4.16
.94
6.32

(22)
547
197
685
678
456
393
525
240
152

(23)
⅛+⅓=

(24)
4.0125
1.5907
4.10
8.673

(25)
⅜+⅝+⅞+⅛=

(26)
12½
62½
12½
37½

(27)
⅛+¼+½=

(28)
¾+¼=

(29)
4¾
2¼
5¼

(30)
2½
6⅜
3¾

(31)
113.46
49.6097
19.9
9.87
.0086
18.253
6.04

(32)
¾+½+¼=

(33)
.49
.28
.63
.95
1.69
.22
.33
.36
1.01
.56
.88
.75
.56
1.10
.18
.56

(34)
⅛+⅜=

(35)
2 ft. 6 in.
3 ft. 5 in.
4 ft. 9 in.

(36)
2 yr. 5 mo.
3 yr. 6 mo.
4 yr. 9 mo.
5 yr. 2 mo.
6 yr. 7 mo.

(37)
16⅓
12⅛
21½
32¾

(38)
25.091+100.4+25+98.28+19.3614=

Series A—Subtraction Scale

By Clifford Woody

Name .....

When is your next birthday? .....

How old will you be? ..... Are you a boy or girl? .....

In what grade are you? .....

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
8	6	2	9	4	11	13	59	78
5	0	1	3	4	7	8	12	37

(10)	(11)	(12)	(13)	(14)	(15)	(16)
7—4=	76	27	16	50	21	270
	60	3	9	25	9	190

(17)	(18)	(19)	(20)	(21)
393	1000	567482	$2\frac{3}{4}-1=$	10.00
178	537	106493		3.49

(22)	(23)	(24)	(25)	(26)
$3\frac{1}{2}-\frac{1}{2}=$	80836465	$8\frac{7}{8}$	27	4 yds. 1 ft. 6 in.
	49178036	$5\frac{3}{4}$	$12\frac{5}{8}$	2 yds. 2 ft. 3 in.

(27)	(28)	(29)
5 yds. 1 ft. 4 in.	10—6.25=	$75\frac{3}{4}$
2 yds. 2 ft. 8 in.		$52\frac{1}{4}$

(30)	(31)	(32)
9.8063—9.019=	7.3—3.00081=	1912 6 mo. 8 da.
		1910 7 mo. 15 da.

(33)	(34)	(35)
$\frac{5}{12}-\frac{2}{10}=$	$6\frac{1}{8}$	$3\frac{7}{8}-1\frac{5}{8}=$
	$2\frac{7}{8}$	

Series A—Multiplication Scale

By Clifford Woody

Name .....

When is your next birthday? .....

How old will you be? ..... Are you a boy or girl? .....

In what grade are you? .....

(1) $3 \times 7 =$

(2) $5 \times 1 =$

(3) $2 \times 3 =$

(4) $4 \times 8 =$

(5) 23 × 3 =

(6) 310 × 4 =

(7) $7 \times 9 =$

(8) 50 × 3 =

(9) 254 × 6 =

(10) 623 × 7 =

(11) 1036 × 8 =

(12) 5096 × 6 =

(13) 8754 × 8 =

(14) 165 × 40 =

(15) 235 × 23 =

(16) 7898 × 9 =

(17) 145 × 206 =

(18) 24 × 234 =

(19) 9.6 × 4 =

(20) 287 × .05 =

(21) 24 × 2½ =

(22) $8 \times 5\frac{3}{4} =$

(23) $1\frac{1}{4} \times 8 =$

(24) 16 × 2⅝ =

(25) $\frac{7}{8} \times \frac{3}{4} =$

(26) 9742 × 59 =

(27) 6.25 × 3.2 =

(28) .0123 × 9.8 =

(29) $\frac{1}{8} \times 2 =$

(30) 2.49 × 36 =

(31) $\frac{12}{25} \times \frac{15}{32} =$

(32) 6 dollars 49 cents × 8 =

(33) $2\frac{1}{2} \times 3\frac{1}{2} =$

(34) $\frac{1}{2} \times \frac{1}{2} =$

(35) 987¾ × 25 =

(36) 3 ft. 5 in. × 5 =

(37) $2\frac{1}{4} \times 4\frac{1}{2} \times 1\frac{1}{2} =$

(38) .0963⅛ × .084 =

(39) 8 ft. 9½ in. × 9 =

Series A—Division Scale

By Clifford Woody

Name .....

When is your next birthday? .....

How old will you be? ..... Are you a boy or girl? .....

In what grade are you? .....

(1) $3\overline{)6}$ (2) $9\overline{)27}$ (3) $4\overline{)28}$ (4) $1\overline{)5}$ (5) $9\overline{)36}$ (6) $3\overline{)39}$

(7) $4 \div 2 =$ (8) $9\overline{)0}$ (9) $1\overline{)1}$ (10) $6 \times ____ = 30$ (11) $2\overline{)13}$

(12) $2 \div 2 =$ (13) $4\overline{)24 \text{ lbs. } 8 \text{ oz.}}$ (14) $8\overline{)5856}$ (15) $\frac{1}{4}$ of $128 =$

(16) $68\overline{)2108}$ (17) $50 \div 7 =$ (18) $13\overline{)65065}$ (19) $248 \div 7 =$

(20) $2.1\overline{)25.2}$ (21) $25\overline{)9750}$ (22) $2\overline{)13.50}$ (23) $23\overline{)469}$

(24) $75\overline{)2250300}$ (25) $2400\overline{)504000}$ (26) $12\overline{)2.76}$ (27) $\frac{7}{8}$ of $624 =$

(28) $.003\overline{).0936}$ (29) $3\frac{1}{2} \div 9 =$ (30) $\frac{3}{4} \div 5 =$ (31) $\frac{5}{4} \div \frac{3}{5} =$

(32) $9\frac{5}{8} \div 3\frac{3}{4} =$ (33) $52\overline{)3756}$ (34) $62.50 \div 1\frac{1}{4} =$

(35) $531\overline{)37722}$ (36) $9\overline{)69 \text{ lbs. } 9 \text{ oz.}}$

Limitations. (*a*) Although the scale is intended to determine what types of examples a pupil or class can solve it does not include all of the types that occur. Thus in addition the only example involving the addition of a one and a two-place number is $\frac{17}{2}$, no examples of the more difficult type $\frac{17}{5}$ being given. (*b*) There are not enough examples of any one type to give a fair test of the pupils' ability to solve examples of that type. Thus in the addition scale the pupils' knowledge of the combinations is tested only by $\frac{2}{3}$ and $3+1$. (*c*) The scales give no test of the pupils' speed and accuracy in solving the different types that they are able to solve correctly.

The Courtis Standard Research Tests, Series A and Series B, together with the Woody Scale, give a fairly complete test of ability in the fundamentals. The Woody Scale determines what types the pupils can solve, or, what is more important, what types they can not solve, while the Courtis Tests measure their speed and accuracy in the facts and complete processes with integers. Even this combination is incomplete, however, as it gives no measure of speed in performing the four operations with fractions, common and decimal, and with compound numbers.

The Cleveland-Survey Arithmetic Tests†

Description. This series of fifteen tests, Sets A, B, C, ... O was devised to "show both the complexity of the processes which a given grade can master, and also the number of examples of a given type that can be performed in a specified time."*

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*Judd, Chas. H.—Measuring the Work of the Public Schools.

The time allowances for the several sets are as follows:

Set A	30 seconds	Set I	1 minute
Set B	30 seconds	Set J	2 minutes
Set C	30 seconds	Set K	2 minutes
Set D	30 seconds	Set L	3 minutes
Set E	30 seconds	Set M	3 minutes
Set F	1 minute	Set N	3 minutes
Set G	1 minute	Set O	3 minutes
Set H	30 seconds		

All of the examples in a given set are of the same type and of equal difficulty, so the measure of the pupils' speed in that particular type of example is the number attempted in the given time. The tests follow.

Set A—Addition

1	6	9	0	4	1	7	9	3	2	1	3	6
2	6	5	1	2	3	7	6	0	4	5	8	9
0	3	8	9	7	8	2	1	4	8	0	2	3
7	2	1	9	6	0	5	6	7	9	5	7	1
4	7	0	3	1	2	5	6	7	5	8	6	9
6	9	8	5	4	9	8	0	2	1	3	5	0
4	2	9	7	4	5	7	4	8	0	3	9	2
3	2	3	8	0	2	1	9	6	0	4	1	8
5	0	6	2	4	5	1	6	3	7	9	0	4
7	4	3	1	8	9	0	2	3	4	8	6	5

Set B—Subtraction

9	7	11	8	12	1	9	13	4	12
9	3	6	1	3	0	7	8	3	6
8	11	12	5	10	6	11	15	10	12
0	9	7	1	2	0	7	8	9	4
2	7	13	3	10	1	6	15	4	8
1	5	7	2	5	1	3	9	2	3
4	10	13	10	9	5	8	17	6	11
4	7	5	1	4	5	6	9	4	8
5	12	15	5	16	7	8	16	9	11
0	9	6	3	8	0	5	7	1	4

Set C—Multiplication

3	4	9	0	5	4	2	7	4	9
2	7	8	2	6	1	9	6	0	5
9	5	4	7	6	2	3	9	0	7
1	2	8	0	5	1	3	6	5	4
1	2	7	0	8	7	3	9	2	4
6	8	7	6	3	1	8	9	0	3
1	4	8	0	4	1	6	8	0	9
5	4	9	3	5	4	2	8	7	3
1	3	6	0	3	2	6	7	5	4
7	4	8	0	9	2	3	9	5	6

Set D—Division

3)9	4)32	6)36	2)0	7)28	9)9	3)21
6)48	1)1	5)10	2)6	4)24	7)63	6)0
8)32	1)8	5)30	8)72	1)0	9)36	1)7
2)10	7)42	1)1	6)18	3)6	4)20	7)49
1)3	2)8	6)6	3)27	8)64	1)2	4)16
5)0	3)24	9)63	2)4	8)24	7)7	2)18
6)42	3)0	7)21	4)4	3)15	9)81	7)0

Set E—Addition

5	2	9	2	6	1	4	9
2	8	8	8	3	4	6	7
2	8	0	5	4	2	5	1
0	5	7	0	8	5	3	5
4	1	6	6	8	4	4	3

6	2	6	8	5	4	1	3
7	7	2	5	9	0	4	7
8	3	3	1	6	8	1	2
5	4	9	3	3	5	8	9
5	1	3	8	8	5	4	6

Set F—Subtraction

616	1248	1365	1092	716
456	709	618	472	344
1267	1335	707	816	1157
509	419	277	335	908
1355	908	519	1236	1344
616	258	324	908	818
1009	768	1269	615	854
269	295	772	527	286

Set G—Multiplication

2345	9735	8642	6789	2345
2	5	9	2	6
9735	2468	6789	3579	2468
9	3	6	3	7
5432	9876	8642	3579	9876
4	8	5	7	4
5432	3689	2457	9863	7542
8	5	6	4	7

Set H—Fractions

$\frac{3}{5}+\frac{1}{5}=$	$\frac{6}{9}-\frac{4}{9}=$	$\frac{4}{9}+\frac{1}{9}=$	$\frac{8}{9}-\frac{7}{9}=$
$\frac{1}{9}+\frac{5}{9}=$	$\frac{3}{7}-\frac{1}{7}=$	$\frac{1}{7}+\frac{4}{7}=$	$\frac{6}{7}-\frac{2}{7}=$
$\frac{2}{9}+\frac{4}{9}=$	$\frac{4}{5}-\frac{1}{5}=$	$\frac{5}{8}+\frac{1}{8}=$	$\frac{6}{9}-\frac{5}{9}=$
$\frac{7}{9}+\frac{1}{9}=$	$\frac{5}{7}-\frac{2}{7}=$	$\frac{5}{9}+\frac{2}{9}=$	$\frac{8}{9}-\frac{1}{9}=$
$\frac{1}{8}+\frac{3}{8}=$	$\frac{6}{8}-\frac{1}{8}=$	$\frac{2}{7}+\frac{1}{7}=$	$\frac{5}{9}-\frac{1}{9}=$
$\frac{2}{9}+\frac{6}{9}=$	$\frac{4}{8}-\frac{3}{8}=$	$\frac{4}{7}+\frac{2}{7}=$	$\frac{7}{9}-\frac{5}{9}=$

Set I—Division

$4\overline{)55424}$	$7\overline{)65982}$	$2\overline{)58748}$	$5\overline{)41780}$
$9\overline{)98604}$	$6\overline{)57432}$	$3\overline{)82689}$	$6\overline{)83184}$
$8\overline{)51496}$	$9\overline{)75933}$	$8\overline{)87856}$	$4\overline{)38968}$

Set J—Addition

7	9	4	7	2	9	6	7	7	8	9	4	3	2
5	2	5	1	9	6	9	1	8	0	5	3	1	1
4	4	8	9	4	2	6	5	5	7	3	7	7	6
2	8	1	4	8	4	7	1	4	1	4	7	6	6
6	2	4	3	5	7	0	4	1	8	6	0	9	1
0	7	8	2	1	1	4	6	8	5	2	2	6	8
5	5	5	8	5	3	3	5	2	1	3	9	3	6
1	3	1	5	2	9	7	3	1	3	9	5	4	9
8	6	3	2	4	2	1	3	3	7	2	6	5	7
3	1	9	7	3	3	6	7	9	4	2	3	4	5
2	4	6	7	6	8	0	6	8	9	8	4	2	2
9	8	3	1	7	5	6	1	4	4	5	8	9	2
9	8	5	9	6	5	6	7	5	4	6	8	9	4

Set K—Division

21)273	52)1768	41)779	22)462	31)837
42)966	23)483	72)1656	81)972	73)1679
21)294	62)1984	31)527	52)2184	41)984
32)384	51)2397	82)1968	71)3692	22)484
41)1681	33)693	61)1586	53)1166	31)496

Set L—Multiplication

8246	3597	5739	2648
29	73	85	46
4268	7593	6428	8563
37	64	58	207

Set M—Addition

7493	8937	8625	2123	5142	3691
9016	6345	4091	1679	0376	4526
6487	2783	3844	5555	4955	7479
7591	4883	8697	6331	9314	2087
6166	1341	7314	6808	5507	8165
5226	9149	6268	9397	7337	8243
2883	8467	7725	6158	2674	6429
2584	0251	8331	3732	9669	9298
0058	7535	5493	4641	5114	7404
2398	5223	3918	7919	8154	2575

Set N—Division

67)32763	48)28464	97)36084	59)29382
78)69888	88)34496	69)40296	38)26562

Set O—Fractions

$11\frac{1}{15} + \frac{1}{6} =$	$\frac{9}{14} - \frac{1}{4} =$	$\frac{3}{5} \times \frac{5}{6} =$
$\frac{5}{6} - \frac{2}{21} =$	$\frac{5}{6} \times 19\frac{1}{20} =$	$11\frac{1}{12} \div \frac{5}{8} =$
$\frac{1}{6} \times \frac{3}{10} =$	$\frac{5}{6} \div 11\frac{1}{15} =$	$5\frac{1}{12} + \frac{2}{8} =$
$20\frac{1}{21} \div \frac{1}{6} =$	$\frac{3}{4} + \frac{3}{18} =$	$\frac{3}{8} - \frac{3}{10} =$

Limitations. (*a*) The test does not include sets of examples in decimal fractions or compound numbers, therefore gives no measure of the pupils' mastery of the fundamental processes with these. (*b*) The number of tests in common fractions (two) is entirely inadequate.

The Monroe Diagnostic Tests

There are twenty-one tests in this series which "furnish a reasonably complete diagnosis of the abilities of pupils to do the operations of arithmetic with the exception of the types of examples involving mixed numbers and integers with fractions."* The tests are too long to be reprinted here.

Summary

From the preceding discussion it is seen that either a combination of the Courtis Tests with the Woody Scale, the Cleveland Survey Tests, or the Monroe Diagnostic Tests will enable the teacher to make a fairly complete diagnosis of the pupils' abilities in the four fundamental processes.

Standard Practice or Drill Tests

The standard tests so far described are not drill devices to be used daily but are intended to be used only occasionally to measure the results of drill work done in other ways. Recently there have been published several devices for daily drill that in themselves provide definite standards of achievement and methods of measuring results and

*Monroe, W. S.—Measuring the Results of Teaching.

diagnosing difficulties. Among the best of these are the Courtis Standard Practice Tests. There are forty-eight different cards in the complete set designed to cover all types of examples in the four fundamental processes with integers. The examples on any one card are all of the same type, thus Lesson No. 1 is on addition examples of the type $\begin{array}{r} 6 \\ 3, \\ \hline 1 \end{array}$ $\begin{array}{r} 8 \\ 4; \\ \hline 7 \end{array}$ Lesson No. 2 on subtraction of the type $\begin{array}{r} 19 \\ 6 \\ \hline \end{array}$ $\begin{array}{r} 32 \\ 7 \\ \hline \end{array}$; Lesson No. 3 on multiplication of the type $\begin{array}{r} 70 \\ 3 \\ \hline \end{array}$ $\begin{array}{r} 84 \\ 2 \\ \hline \end{array}$ and Lesson No. 4 on division of the type $1\overline{)76}$, $3\overline{)129}$. Each process recurs at intervals, a new difficulty being introduced each time until the pupil has mastered all of the difficulties involved in the process. The number of examples on each card is such that the time for all of the cards is the same in a given grade. In the low fourth the time allowance is 6¼ minutes. The first time the tests are used each pupil is supplied with a copy of Lesson 13, which is a test card on addition, subtraction, multiplication and division, and covering all of the points involved in the first twelve cards. The card is placed under a transparent sheet of paper and the pupil writes his results on this paper. As a result of this device the cards can be used over and over again until worn out. The pupils start at a given signal and stop at a signal at the end of the given time. Those having perfect papers are excused from the drill work on the first twelve lessons and spend their time on other work assigned by the teacher. The other pupils, the next day, start on Lesson No. 1, and repeat it each day until they get all of the examples correct in the given time, when they go on to No. 2, etc.

A drill device such as this, possesses many advantages. (1) Provides for individual differences; each pupil gets the amount of drill necessary on each card to bring him

to the standard and no more. As the time allowance is the same for all cards different pupils can be working on different cards at the same time. (2) Prevents waste of time by unnecessary drill. When a pupil completes all of the cards for his grade he devotes his time to something else. (3) Diagnoses difficulties of class and individuals. Because of the careful analysis and separation of difficulties the teacher is enabled to locate and overcome weaknesses exposed by the tests.

The Studebaker Economy Practice Exercises in Arithmetic are similar to the Courtis Practice Tests and so will not be described here.

Standard Scores

The Median. In most cases the standard scores here given are median scores. The *median* is a measure of a class or group somewhat similar to but differing from the average. It is the middle score, the score that just as many pupils exceed as fall below. If in a class of seven pupils the individual scores on the Courtis Test in Addition, Series B, were 5, 5, 6, 7, 8, 8, and 20, respectively, the median score would be 7, as three pupils solve more and three less than this number of examples. The median is usually taken as the measure of the class rather than the average, because it is easier to figure and is more typical of the group as a whole, as it is less affected by extremely good or bad scores. Thus in the class given above the median is 7, while the average is $8\frac{3}{7}$. The median is more typical of the class than the average, which is unduly influenced by the one exceptionally good score of 20.

The median scores here given do not necessarily indicate what a pupil should be able to do, or what he needs to be able to do in order to get along in this world. Being derived from results actually obtained in our schools they simply indicate the results of our present teaching. It is

possible that these results might be improved by better teaching. *Ultimately we should have standards based upon what the world demands,* but until we get such, the standards we have, based upon present results, are much better than none at all.

COURTIS STANDARD RESEARCH TESTS.

Median and Standard June (End of Year) Scores.

Series A. Time, 1 Minute

Grade	Test No. 1 Addition	Test No. 2 Subtraction	Test No. 3 Multiplication	Test No. 4 Division
3	xx 27 26	xx 18 19	xx 16 16	xx 15 16
4	42 35 34	30 26 25	29 26 23	27 22 23
5	50 42 42	37 33 31	35 31 30	35 28 30
6	57 49 50	41 39 38	38 37 37	40 35 37
7	62 56 58	46 44 44	41 42 41	45 42 44
8	70 63 63	52 48 49	46 46 45	51 47 49

TEST NO. 7. FUNDAMENTAL PROCESSES.

Grade	Attempts			Rights		
3	xx	4.9	5.0	xx	0.8	1.7
4	8.8	6.8	7.0	4.2	2.5	3.5
5	10.9	9.0	9.0	5.8	4.7	5.2
6	12.5	10.5	11.0	7.0	6.3	6.7
7	14.0	11.7	12.5	8.5	7.2	8.2
8	15.7	13.7	14.0	10.1	8.6	9.4

The first figure in each case is the median New York score based on 27,000 cases. The tests were not given in the third grade in New York. The second figure is the median Boston score based on 18,000 cases. The last score is the standard proposed by Mr. Courtis based on more than 60,000 children.

Series B

Grade	Addition Time, 8 Min. *	**	Subtraction Time, 4 Min. *	**	Multiplication Time, 6 Min. *	**	Division Time, 8 Min. *	**
4	7.4	64	7.4	80	6.2	67	4.6	57
	6	100	7	100	6	100	4	100
5	8.6	70	9.0	83	7.5	75	6.1	77
	8	100	9	100	8	100	6	100
6	9.8	73	10.3	85	9.1	78	8.2	87
	10	100	11	100	9	100	8	100
7	10.9	75	11.6	86	10.2	80	9.6	90
	11	100	12	100	10	100	10	100
8	11.6	76	12.9	87	11.5	81	10.7	91
	12	100	13	100	11	100	11	100

*Speed. **Per cent accuracy.

Speed is the number of examples done in a given time. Accuracy is obtained by dividing the number correct by the number attempted. The upper figure in each case is the median score based upon "many thousands of individual scores in tests given May or June, 1915-16." The lower figure is the standard proposed by Courtis after three years' use of these tests.

Woody Scale

The Woody Scale is too recent for any standards to be available.

The following tentative standards are proposed by Mr. Woody, based on actual achievements of the children tested with the preliminary tests that were used in deriving the scale as it now stands. As already indicated, the examples

in each scale are of unequal difficulty and the relative difficulty or "value" of each example was determined by Mr. Woody and is given in the following table:

Established Value of Each Problem in the Woody Scales

No. of Problem	Addition	Subtraction	Multiplication	Division
1	1.23	1.06	0.87	1.57
2	1.40	1.48	1.05	2.08
3	2.50	1.50	1.11	2.18
4	2.61	1.50	1.58	2.31
5	2.83	1.70	2.38	2.40
6	3.21	1.75	2.62	2.46
7	3.26	2.18	2.68	2.56
8	3.35	2.51	2.71	3.05
9	3.63	2.57	3.78	3.16
10	3.78	2.65	3.79	3.20
11	3.92	2.88	4.09	3.49
12	4.18	2.90	4.26	3.59
13	4.19	2.96	4.71	3.96
14	4.85	3.64	4.72	4.06
15	4.97	3.70	4.73	4.60
16	5.52	4.35	5.05	4.67
17	5.59	4.41	5.20	4.98
18	5.73	4.42	5.24	5.16
19	5.75	5.18	5.38	5.26
20	6.10	5.52	5.63	5.31
21	6.44	5.70	5.72	5.36
22	6.79	5.75	5.83	5.48
23	7.11	5.76	5.83	5.56
24	7.43	5.91	5.89	5.58
25	7.47	6.77	6.29	5.78

No. of Problem	Addition	Subtraction	Multiplication	Division
26	7.61	7.07	6.30	5.91
27	7.62	7.21	6.58	6.04
28	7.67	7.38	6.85	6.43
29	7.71	7.41	6.97	6.76
30	7.71	7.41	7.00	6.83
31	7.97	7.49	7.07	6.87
32	8.04	7.52	7.07	6.88
33	8.18	7.69	7.29	7.22
34	8.22	7.72	7.50	7.24
35	8.58	7.84	7.65	8.17
36	8.67		7.66	8.23
37	8.67		8.02	
38	9.19		8.53	
39			8.61	

The score of the class is taken as the degree of difficulty which an example must possess to be correctly solved by just 50 per cent of the class. The following table gives the tentative standards proposed by Mr. Woody.

Tentative Standards for Series A—Woody Scale

Grade	Addition	Subtraction	Multiplication	Division
2	3.12	1.44		
3	4.99	2.96	1.89	2.54
4	6.11	4.22	4.05	3.21
5	6.99	5.47	5.53	4.94
6	7.95	6.46	6.72	5.87
7	8.65	7.31	7.26	6.59
8	9.01	7.61	7.93	7.16

These standards give the degree of difficulty that an example must possess in order that just 50 per cent of the class can solve it correctly. Thus, if a problem in addition

has 3.12 units of difficulty it should be solved by 50 per cent of the second grade; if it has 4.99 units of difficulty it should be solved by 50 per cent of the third grade, etc.

Cleveland-Survey Tests

There are no standards available as yet for these tests. They have been given, however, in Cleveland, Grand Rapids and St. Louis and the median scores from these cities, although they should not be regarded as standards, give some idea of the results we can expect to obtain.

Cleveland-Survey Tests. Cleveland and Grand Rapids Scores (Median Number of Examples Correct)

The upper number in each case is the median score obtained in Cleveland, the lower number the median obtained in Grand Rapids. In all cases the medians are for the lower half of the grades.

Test	Grade 3b	Grade 4b	Grade 5b	Grade 6b	Grade 7b	Grade 8b
A	13.4	17.8	22.2	24.8	26.7	27.5
	11.8	13.6	20.3	22.8	26.5	29.5
B	9.3	13.4	17.2	19.8	21.5	26.0
	6.3	9.1	14.7	16.8	21.3	22.8
C	6.5	12.0	15.5	16.6	17.7	19.0
		7.1	13.7	15.5	17.7	19.3
D	6.3	12.4	15.7	18.5	20.8	22.5
		6.9	12.5	15.5	18.4	20.5
E	4.3	5.3	6.3	6.8	7.5	7.8
		4.1	5.2	6.0	7.2	7.8
F	2.0	4.9	6.7	7.5	8.6	10.1
		2.8	6.0	7.1	9.3	10.3
G	2.0	3.9	5.2	5.5	5.9	6.6
		2.2	4.5	5.3	6.1	6.7
H	0.0	0.0	5.0	5.5	7.7	8.5
				6.2	9.0	8.6

Test	Grade 3 b	Grade 4 b	Grade 5 b	Grade 6 b	Grade 7 b	Grade 8 b
I	0.6	1.1	2.0	3.1	4.0	4.7
		0.7	1.3	2.3	3.8	4.0
J	1.9	3.2	4.0	4.1	4.9	5.7
			3.4	4.1	5.4	5.7
K	0.0	4.0	6.8	8.5	10.1	12.5
			3.0	5.4	7.5	9.7
L	0.0	1.7	2.5	2.8	3.2	3.9
			2.3	3.3	4.3	4.9
M	1.4	2.5	3.2	3.8	4.4	5.1
			3.0	4.3	4.9	5.7
N	0.0	0.8	1.3	1.7	2.0	2.6
			0.7	1.1	1.7	2.0
O	0.0	0.0	0.0	3.1	4.1	5.5
				3.5	3.9	5.5

Cleveland-Survey Tests. St. Louis Scores

(Median Number of Examples Correct)

The following table gives the median scores obtained in St. Louis. The first figure in each case is for the lower half of the grade, the second figure for the upper half.

Set	Grade 3		Grade 4		Grade 5		Grade 6		Grade 7		Grade 8	
A	14.6	18.3	19.8	21.3	22.5	22.5	26.3	26.4	27.8	28.4	32.3	32.2
B	9.9	12.2	17.1	17.0	18.0	20.0	20.3	20.6	22.8	24.2	26.7	28.3
C	7.6	10.5	16.7	15.4	16.9	16.7	18.2	18.3	18.9	19.8	20.7	21.9
D	9.0	12.2	15.8	16.3	18.4	17.8	19.3	20.5	21.3	22.3	23.8	25.7
E	3.8	4.8	5.7	5.4	6.0	6.1	6.9	7.1	6.6	7.4	8.0	8.4
F	2.3	3.5	5.6	6.0	6.4	7.4	8.0	8.3	8.5	9.6	10.1	11.3
G	2.7	3.5	4.9	5.1	5.5	5.6	5.9	6.2	6.4	6.9	7.4	7.8
H	0.7	3.8	7.8	6.8	4.8	6.5	8.0	8.1	9.5	9.7	10.8	12.0
I	1.1	1.4	2.0	2.0	3.0	3.2	3.9	4.1	4.5	5.0	5.4	5.8
J	1.6	2.9	3.8	3.9	4.1	4.3	5.0	5.1	5.2	5.3	5.4	5.8
K	0.0	0.0	3.3	4.0	5.0	5.8	6.9	7.4	8.3	9.7	10.3	11.7
L	0.0	0.0	2.5	2.9	3.1	3.4	4.3	4.1	4.6	4.7	5.2	5.3
M	0.5	2.1	2.9	3.3	3.4	3.7	4.2	4.4	4.5	4.9	5.2	5.3
N	0.0	0.0	0.8	1.1	1.3	1.4	1.6	1.8	2.0	2.0	2.6	2.7
O	0.0	0.0	0.0	2.5	3.3	3.3	3.6	4.1	4.8	5.6	6.1	6.6

PART IV

DEVELOPING THE ABILITY TO APPLY THE FUNDAMENTALS OF ARITHMETIC TO CONCRETE SITUATIONS

CHAPTER I

THE PURPOSE OF THE PROBLEM WORK IN ARITHMETIC

Purpose. Important as it is, the mechanical mastery of the fundamental facts and processes is not the ultimate end of arithmetical instruction but only a means to that end. The ultimate aim is to prepare the boys and girls for life, and life demands not only a perfect mastery of the fundamentals of arithmetic but also the ability to apply these fundamentals successfully to actual life situations. No matter what the pupil's life work may be, farming, business, home-making, teaching, a mechanical trade, engineering, the law, or medicine, it will present situations having a numerical side, and it is the duty of the school to prepare its pupils to meet such situations successfully. In order to do this these life situations, or their descriptions, are brought into the school and constitute the applied problems of arithmetic. The chief purpose of these problems is to prepare the pupils to solve the problems they may meet in life. There are two ways in which we might try to do this. (1) Give direct preparation for the kinds of problems the pupils will meet in life, and (2) Give general preparation for all kinds of problems.

Direct Preparation for the Kinds of Problems the Pupils Will Meet

In order to give direct preparation for the kinds of problems that the pupils will meet in life it would be necessary to look ahead and to make a collection of all of the different types of these problems. Having done this the pupils could be given a rule or a type solution for each of these types and drilled in using it until they could solve all such problems mechanically without having to stop and think them out. This would be the surest method of preparing the pupils to solve the problems they will meet in life if it were possible. The only trouble with it is that it is not possible for several reasons.

In the first place no one can foresee what types of problems the pupils of a given class are going to meet in their lives. This will depend largely on the character of their life work and no one can predict what that will be. Even in a rural, agricultural community it is not safe to presume that the boys are all going to be farmers and the girls farmers' wives. Even if the teacher knows the lines of work in which her pupils will engage it is not always possible to predict the kind of problems they will have to solve. Conditions in any given field of work are constantly changing and these changes often take place in a short time and are revolutionary in character. The arithmetic of business ten years from today may be quite different from what it is today.

In the second place it is not possible to classify all of the problems that occur in life according to a few types and to formulate rules or type solutions for each. Indeed, it may almost be said that in order to have enough types to "go around" there would have to be as many types as there are individual problems. It is not possible to prepare the pupils directly for all of the types of prob-

lems they may meet in life. Even if the schools could teach enough types they would not have sufficient time to drill on them and fix them permanently in the pupils' minds, and as a result they would certainly be forgotten in a few days, months or years.

Another difficulty arises in recognizing the rule or type solution to be applied to a given problem. Confronted by an arithmetical situation in life the pupil might fail utterly, not because he had never been taught the proper solution, not because he had forgotten it, but simply because he could not tell which of his various rules or type solutions to apply to this particular problem.

Conclusions. The teacher who relies on rules and type solutions to enable her pupils to solve the arithmetical problems of life is pre-doomed to failure, she can not possibly teach enough types to cover all of the problems her pupils will meet, she has not sufficient time to make certain that the pupils will retain the types she does teach, and there is always the danger that the pupils will apply the wrong type to a given problem. In spite of the utter futility and hopelessness of solving problems by rule or type solution many teachers consciously or unconsciously use the method. As a result many, if not the majority, of the pupils graduated from the grades, when they meet an arithmetical problem, try to remember how they worked problems of that kind in school. If they forget or never worked a problem similar to the given one they are utterly lost. Such pupils have no mastery over problems in general and are very poorly prepared for their life work, whatever it may be.

General Preparation for All Kinds of Problems

Although many high school and college students can solve only those types of applied problems in arithmetic that they have been taught to solve in the grades, there are

many others who can solve almost any problem of a reasonable degree of complexity. Upon investigation it will usually be found that these students attack a problem in an entirely different way; instead of trying to remember how the teacher or the book solved a problem like the given one yesterday or the day before, or how they themselves worked similar problems at some previous time, they size up the given problem on its own merits, get clearly in mind the relations that exist between the various quantities involved and think out the solution in terms of these relations. These students have developed the power to successfully cope with any kind of an arithmetical situation they may encounter in life and the school can develop this same power in all or at least the majority of the pupils in the same way that it was developed in these students.

Mastery of problems involves the ability to plan the solution as well as the ability to execute the plan successfully. This ability to plan the solution does not come through the blind following of rules or directions; it can only come from meeting many different kinds of problems and reasoning each through in terms of the relationships involved. Probably everyone has had experience with some workman who could do fairly good work as long as some one told him what to do, but was totally unable to go ahead for himself—who had never formed the habit of thinking and planning for himself but relied on others for directions. The pupil who relies on type solutions and rules in arithmetic is in the same condition as this workman and will never get a real mastery over problems until he forms the habit of thinking and planning them out for himself.

By thoughtfully solving a variety of problems beginning with the easiest the power to think out other and more difficult problems is gradually developed. If the teacher has her pupils think out each problem as it occurs the

majority of them will eventually develop power over problems in general and will be well equipped for the problems they will meet in life. When the pupil solves a problem by mechanically following some set of directions or type solution he gets nothing but the answer to that particular problem; but every time he reasons out a problem he not only gets the result to that particular problem but adds to his power to reason out other problems. In order to realize the social aim of arithmetic the fundamental facts and processes must be mechanized from the first but the solution of problems must always be thoughtful and not mechanical. To let the solution of problems become mechanical and to let the work with the fundamentals become thoughtful are equally bad; either will defeat the ends sought.

Only one exception need be made to this general proposition. If a certain type of problem is of such common and frequent occurrence in life that all of the pupils are sure to meet it frequently then the teacher is justified in formulating a rule and mechanizing its solution. If this is done, however, she must provide sufficient drill, spread over a sufficient interval of time to insure permanent retention. An example of this kind is found in simple interest. This particular application of arithmetic occurs so frequently in the lives of most people that the schools should reduce the figuring of simple interest to a mechanical basis and drill on it for several years (at least from the sixth to the eighth grade) to insure its retention.

It may be thought by some that teachers no longer teach their pupils to solve problems by rule. A little observation in almost any school system is sufficient, however, to convince the most skeptical that there are still many teachers who do so, consciously or unconsciously. Many teachers apparently have no aim in problem work beyond teaching in such a way as to enable their pupils to solve the prob-

lems in the book, and the easiest way to accomplish this end is to give a rule or type solution. The author has seen many examples of this sort of problem work within the last five years. The worst was a recitation given in a seventh grade. The pupils had been assigned a list of ten applied problems in proportion as home work and at the beginning of the period the teacher had the solutions of those problems written on the board. The first problem was, "If A has $1200 invested in a certain business and receives a profit of $130 at the end of the year, what profit should B receive if he has $750 invested in the same business?" The pupil's solution of this problem as written on the board was as follows:

$1200 : $750 :: $130 : x.

$$x = \frac{\$750 \times \$130}{\$1200} = \frac{325}{4} = \$76.75 \text{ B's profit.}$$

After the solutions had all been written on the board the pupils proceeded to explain them. The first problem was explained as follows, partly by the pupil himself and partly in response to the teacher's questions, "The quantity to be found is the number of dollars of B's profit, the quantity having the same name is $130 the number of dollars of A's profit, so we place $130 in the third term of our proportion. B's profit will be smaller than A's because he has less money invested so we place the smaller of the other two quantities, $750, in the second term of our proportion and the larger, $1200, in the first term. Then x is the second term multiplied by the third and divided by the first. So x equals $750 times $130 over $1200 or $76.75 B's profit. All of the other problems were explained in the same way. It was evident before the recitation was half over that the pupils were solving the problems by applying the following rule which they had evidently

memorized. *Rule:* Place the quantity having the same name as the quantity to be found in the third term and the quantity to be found or x in the fourth. Then, if the quantity to be found is to be larger than the quantity in the third term, place the larger of the other two quantities in the second term and the smaller in the first; but, if the quantity to be found is smaller than the quantity in the third term, place the smaller of the other two quantities in the second term and the larger in the first. Then x, or the quantity to be found, will be equal to the product of the second and third terms divided by the first.*

Other problems were then solved in class. The first step was always to put down the dots as follows : :: : . Then, if the pupils had difficulty, the teacher helped them by asking "What do we always put in the fourth term?" (*Ans.* x), "What do we put in the third term?" (*Ans.* The quantity that has the same name as the quantity we want to find), etc. In this way they proceeded to fill in the holes between the dots until they had built up the proportion.

If the sole purpose of this recitation was to enable the pupils to solve the problems in proportion in their text it was entirely successful. They solved all of the problems without any serious difficulty. But if the purpose of the recitation was to enable the pupils to use proportion as a tool in solving problems in after life it was an absolute failure.

Not a single pupil in the class had any idea of what a proportion is, not one of them had any understanding of the relationship existing between the four quantities involved, and as a result not one of them would recognize a situation in life that could be solved by means of proportion. They only knew that these problems were to be

*A similar rule is found in all of our old arithmetics and is known as the "Rule of Three." It is an earlier form of proportion.

solved by this particular rule because the book or teacher said so. If proportion is worth teaching at all the important thing that the pupils must get is a grasp of the relationships involved. If they get this they will recognize a situation in which they can use proportion when they meet it and will not need a rule to enable them to build up the proportion. In the problem given above, if it is solved by proportion, the important thing is for the pupils to see that B's profit will bear the same relation to his investment, as A's profit bears to A's investment; or that B's profit will bear the same relation to A's profit as B's investment bears to A's investment. Seeing these relations the pupils can form the proportion without the aid of a rule.

CHAPTER II

THE NATURE AND SOURCES OF PROBLEMS

Essentials of Good Problems. The three most important requirements to be satisfied by problems are (1) They must be concrete to the pupils; (2) They must be typical of real life, and (3) They must be varied.

Problems Must Be Concrete to the Pupils

The first requirement for any problem is that it be concrete to the pupils who try to solve it, that is, it must be about things and situations that come within the pupils' own experience and consequently mean something to and are understood by them. Unless this is true the pupils can not think through the situation and are driven to some mechanical method of solution. Even an adult can not successfully think through a situation foreign to his experience. This is illustrated by the following account by a normal school teacher: "I was trying to explain to a teacher of several years' experience that the quotient of the length of a row, divided by the distance between plants, did not give the number of plants in the row. * * * After I had exhausted my stock of word pictures, I took a pencil and began a diagram, when the teacher to whom I was explaining the problem, exclaimed: 'Oh, I see it. It is just like hanging handkerchiefs on a clothesline; it takes one more clothespin than there are handkerchiefs.' "* The two situations were geometrically identical, but the teacher could not visualize the situation "plants in a row" and could visualize "handkerchiefs on a clothesline" be-

*Quoted by Dr. J. F. Millis in "The Solution of Problems in Arithmetic." (Pamphlet.)

cause she had had experience with the one situation and had not with the other. Several years ago in a class in solid geometry the author assigned a list of applied problems on the volume of the cone. Several members of the class failed to solve one of the problems, although it was no more difficult than the rest, simply because the dimensions were given in centimeters whereas in the other problems they were given in feet and inches. In this school solid geometry was taught in the third year and physics in the fourth and some of the pupils did not know what centimeter meant and this unfamiliar element was sufficient to prevent them solving the problem. *If the pupils are to think through a problem the situation involved must be concrete to them so that they can image the relationships between the quantities.* It must be remembered that a problem is not necessarily concrete just because it has to do with concrete things. Indeed "concreteness" is not an intrinsic quality, a quality inherent in the problem itself. A given problem may be concrete to one pupil and not to another according to the pupils' individual experience.

A problem has been defined as (1) an actual life situation having an arithmetical side; or (2) the description of such a situation. There are thus two types of problems. To illustrate these take the situation or problem of finding the number of board feet in a pile of lumber. This might be presented to the pupils by taking them to the manual training department or to a lumber yard, showing them a pile of lumber and telling them to find the number of board feet in the pile. Or it might be presented to them in words as follows: "A pile of lumber contains ten pieces $2'' \times 4'' \times 10'$; five pieces $2'' \times 6'' \times 10'$; and twenty pieces $1'' \times 5'' \times 10'$. Find the number of board feet in the pile."

Actual Life Situations. The problem met as an actual life situation is both harder and easier than the problem met in words. It is easier because it is more concrete

and so it is easier for the pupils to see the relation of the various quantities involved. It is harder because the successful meeting of the life situation involves an element not encountered in the situation described in words. In the one case the required data is given and all the pupils have to do is to determine how to make use of this information, in the other case the data is not given and the pupils must first decide what information or data they will need and must then get that data for themselves. It follows, therefore, that the pupils can not develop the ability to apply their arithmetic to actual life situations by applying it to the descriptions of such situations. Ideally all of the problems of arithmetic should be met by the pupils as actual life situations. In actual practice we may never reach this ideal but at least we can see to it that the pupils meet many problems in this way and when for any reason the teacher must use the description of an actual life situation she should realize that it is only at best a very poor substitute for the real situation istelf.

Several illustrations of problems met as actual life situations follow: (1) Keeping a personal account of all receipts and expenditures. Each pupil keeps an account of all of the money he receives, giving date and source, and of all he spends, giving date and what it was spent for. Every week he balances his accounts. (2) Keeping books for a paper route. If any of the boys carry papers they can be shown how to keep a simple set of books. (3) Painting a room. Each pupil imagines that he is a painter and bids for the job of painting the school room. To do this the pupils must first take their own measurements, find out how much surface a gallon of the desired paint will cover and the cost per gallon, and figure out the total cost of the paint required. They must also find out about how long it would take a painter to do the job and the cost per hour for his labor, figuring the total cost of labor.

Descriptions of Life Situations. In the past most of the problems we have used have been taken from books or have been made up by the teacher and presented to the pupils *in words*. At best such problems are but the descriptions of actual life situations and at their worst they are the descriptions of hypothetical situations such as never did and never could occur in life.

Many of the graduates of our elementary and high schools today can solve book problems but are utterly unable to solve the problems they meet in life, largely because book problems are the only kind they have ever met in their school work. As an illustration of this the author in his work with normal college students finds that a large per cent of them can not find the area of a triangle. To be sure, if they are given the problem, "The base of a triangle is 3′ 6″ and the altitude 2′ 8″. Find the area," they can solve it, but if they are given a triangular piece of cardboard or taken to a triangular field and asked to find the area many of them are at a total loss how to proceed.

Perhaps one of the weakest points in much of our school work is its artificiality, its abstractness. Every one has heard of the teacher in a Mississippi River town whose pupils studied about the Mississippi River in their geographies and never once connected it with the river they saw every day. This teacher was no worse than the arithmetic teacher who teaches the text book problems on Papering and never has the pupils measure a room and estimate the paper required, who teaches Stocks and Bonds without the pupils ever getting a clear idea of either or of the difference between the two, who teaches Taxes from the book and never considers the local situation, or the teacher who teaches Commercial Discount without considering the discount sales held in the local stores and advertised in the local papers.

Probably the time will never come when the pupils will meet all of their problems in the form of actual, concrete, life situations. The teacher will always have to present many of the problems in words but should realize the weaknesses of such problems. Too often the words of the problem are mere words and convey no idea to the pupils, call up no concrete images in their minds. Teachers too often permit pupils to deal with mere words without making sure that the correct idea is behind the words. They are apt to have too great faith in words, forgetting that they often convey no idea, or entirely the wrong idea to the pupils. The successful teacher must learn to be exceedingly suspicious of words and use every possible means to make sure that the words convey the proper idea to the pupils.

Problems Made by the Pupils and Teacher. Many of the problems used in arithmetic should be made up by the pupils themselves and others should be made up by the teacher. Such problems can be made concrete by basing them on (1) Things the pupils are doing in school, (2) Things the pupils are doing outside of school, and (3) Community activities which either are familiar or can easily be made familiar to the pupils.

Problems Based on What the Pupils Are Doing in School. In every school, from the first grade to the eighth, the pupils are constantly having experiences and meeting situations that should be made the basis of problems for arithmetic. Many teachers are utterly blind to these opportunities and teach arithmetic in the arithmetic period and forget all about it at other times. The author once visited a first grade and saw the teacher direct the pupils to a certain page in their primers by holding the open book before them, pointing to a picture on the opposite page and having the pupils leaf through the book until they found the picture. It afterwards developed that this

teacher was teaching the pupils their numbers, but she had a separate number period for this.

The opportunities for problems may arise in connection with (1) the other subjects studied, (2) the routine of the school, (3) the games and other activities at recess, or (4) some of the special activities of the school. Reading, Language, Spelling, Nature Study, Industrial Arts, Home Economics, Agriculture, History and Geography all present situations having a numerical or quantitative side that can be made the basis of many interesting problems by a wide awake teacher and class. The routine of the school affords many more opportunities such as passing, collecting and taking care of material used, the keeping of a temperature chart of the room and the keeping of absence and tardy records. The keeping of scores in games and the laying out of a playground, or a baseball diamond are examples of problems arising in connection with the play activities of the school. The planning and financing of school entertainments, corn clubs, pig-growing contests and home credit work of all kinds also lead to many problems.

Problems Based on What the Pupils Are Doing Outside of School. Besides knowing what the pupils are doing in connection with their school work the teacher should have some knowledge of what they are doing outside. The city boy may have a paper route, or be selling the Saturday Evening Post; the country boy may be farming an acre on his own account or raising a pig. With only a little encouragement the pupils will bring problems arising from these and other outside activities to the arithmetic class.

Problems Based on Community Activities. In making problems the teacher does not need to confine herself to what the pupils are actually doing themselves. In any community at any time there are many things going on that can easily be brought within the pupils' experience and understanding and that make good problem material.

In the country there are such things as agriculture (the chief occupation of the community), good roads, local finances and taxes, school finances, size of crops, per cent of increase and decrease in acreage devoted to various crops, per cent of increase and decrease in amount of various crops, market quotations on farm crops and produce, etc. In the city the field is also limitless, including study of important local industries, local finances and taxes, school finances, improvement of streets, civic improvements of all kinds, sales held at local stores, etc.

Text Book Problems May be Localized. In making up problems the teacher will often find the problems in some modern text very suggestive. Thus, one of our recent arithmetics has a list of problems on "Planning Journeys." The author happened to see recently two different recitations based on this list of problems. The first teacher had assigned the problems just as they were in the book. The second teacher had taken the idea and made up other problems similar to those in the book but based on local conditions. For example, the second problem in the book was, "A single ticket between two places 79 miles apart costs $1.80; the round-trip ticket $3.00. How much is saved on the round trip by buying the round-trip ticket instead of two single tickets?" The second teacher had in making the assignment the previous day proposed a similar problem to her class, namely how much would they save on a trip to Toledo (the nearest large city) by buying a round-trip ticket. The pupils first obtained the required information from the local ticket office and then solved the problem. Both of these recitations were very good, but of the two the second was much more interesting to the pupils and undoubtedly meant much more to them.

Text Book Problems. Although many of the problems should be met by the pupils in the form of actual life situations and many others should be made up by the

pupils themselves and the teacher, yet in most schools the chief source of problems must always be the text book. The chief objection to the text book problem as such is that it is apt to lack concreteness. A text book is not written for any particular class, for any one town or even for any one state. On the contrary it is written for the whole country and usually for the city school as well as the country school. The result is that any text book contains some problems that are concrete to a particular class of pupils, others that can easily be made concrete and still others that can not possibly be made concrete to that class. The first will give no trouble and the last must be omitted entirely; the teacher's chief problem is with the second group, namely, those that are not concrete to the pupils but can be made so.

Preliminary Discussion and Explanation. The teacher's first duty in using text book problems is to see that each problem is concrete to her pupils, that the words of the problem call up in the pupils' minds the correct ideas and images. Sometimes all that is necessary is a brief discussion of the situations involved in the problems when making the assignment. The author recently visited a sixth grade class that was working with a list of problems in their text based on school athletics. One of the problems introduced the term "High School Athletic Association," and as many of the pupils did not know the meaning of this they were not able to solve the problem. The teacher had one of the pupils who did know what was meant by a High School Athletic Association explain to the others, who then had no further difficulty.

Dramatization. Sometimes the best way to make a problem or group of problems concrete is to dramatize the situation involved. Many of the simple problems of the primary grades are based on buying things because this is at the same time one of the earliest, most common and

most important applications of arithmetic. Many of the pupils, particularly in the villages and towns, will have had actual experience in buying things at stores or will at least have accompanied their parents. To amplify this experience and make sure that it is a common possession of all the pupils is the purpose of Playing Store in school. It is not necessary to play store every day, but by devoting one period a week, or every two weeks, to the store the teacher can make sure that the pupils are provided with a concrete basis for problem work on the other days.

Many problems and lists of problems in our newer arithmetics lend themselves readily to dramatization. Thus, in one arithmetic intended for the third grade there are lists of problems on "Going to Grocery for Mother," "The Three Bears," "Ned and His Rabbits," "Little Bo-Peep," "The Doll Party," etc., and in another book intended for the same grade are problems on "Playing School," "Playing Postman," "Buying Toys," "Playing Clerk," etc., all of which can be made concrete by dramatization.

Although it will probably be necessary to resort to dramatization more frequently in the lower grades than in the upper, this is not because it is less effective in the upper grades but simply because the pupils in these grades have a larger body of knowledge and experience. Dramatization should be used even in the upper grades when dealing with problems based on situations foreign to the children's experience. Ideally, perhaps, it might be better to wait until the pupils get the experience before giving them the problems; but there are some of the most important applications of arithmetic particularly to business that the pupils must get in the upper grades *or they will not get them at all.* The ideal place to teach the business applications of arithmetic such as interest, bank discount, commercial discount, figuring profits in a business, stocks

and bonds, taxes, insurance, etc., would be the senior year of the high school or college, as the students would then have greater business experience as a basis for the work and would also have occasion to use the knowledge gained at once. Most of our pupils, however, never get beyond the elementary school and must get this knowledge there or not at all. This makes it necessary for the teacher to first supply the concrete experience necessary to an understanding of these business applications of arithmetic and the best way to do this in many cases is by dramatizing the situations. As an illustration, in many schools the pupils form a school bank, open accounts, make deposits, write checks, borrow money, make out notes, figure interest and discount notes. This is not only the most effective but also the easiest and quickest way of giving the pupils an insight into these topics.

Unified Lists of Problems. In our older text books the problems were usually organized according to the arithmetic involved. Thus, there is a list of problems on addition, another on subtraction, still another on common fractions, etc. Most of the newer books have many of their problems organized in a different way, the unity in a given list of problems instead of being arithmetical is social, that is, in a list of problems the arithmetic may be varied but the problems are all based on the same social or life situation. Thus, we find lists of problems on "Earning Money," "Sending Money by Mail," "Paying Household expenses," "Household Supplies," "Cooking" and "Sewing."

Such an organization has two decided advantages. In the first place it is true to life. The problems met by a grocer in his work are not all addition problems one day, and subtraction the next, instead on any one day they may be quite varied as to the arithmetic involved but are all connected in that they all arise from the business of

managing a grocery store. In the second place it is much easier to make problems that are grouped around a single social situation concrete to the pupils. If a list of ten problems involves ten entirely different situations, none of them concrete to the pupils, it would take too much time to explain each of the ten situations, but if the ten problems all have the same social setting, time can be taken to study this social situation with the pupils and even to dramatize it if necessary.

Life Problems. Since the whole purpose of solving arithmetical problems in school is to develop the ability to solve the arithmetical problems of life it follows that the problems in school must be of the same kind as the problems of life. Under the domination of the old disciplinary conception of arithmetical instruction the problems of our arithmetics were largely of the puzzle type and any problem was considered satisfactory if it involved the required number relations regardless of whether or not it ever occurred in life. In the past in writing a text book on arithmetic the author got his problems not from a study of life but from other arithmetics. As a result our arithmetics of ten years ago were full of problems inherited from past ages, many of which never did occur in actual life and many others that were once life problems but had long been obsolete.

Within recent years several series of arithmetics have appeared that have been written in an entirely different way. The authors instead of going to other books for their problems enlisted the co-operation of men engaged in many different lines of work and with their assistance collected problems that actually occur in life work. As already pointed out, it is impossible to bring all of the different types of problems that occur in life into the school, but we can at least bring in the most important types, those frequently met by the ordinary man; and

we are certainly not justified in using problems that never occur in life or that are met only by the specialist in some particular line of work.

The term "practical problem" has been so much abused that it is meaningless. Many texts boast of their practical problems because they have lists of problems on "Farming," "Cooking," and "Profit and Loss," but a careful study of these problems will often show that the farming problems were never met by any farmer, the cooking problems by any cook, or the profit and loss problem by any business man. To be practical in the true sense of the word a problem must not only be about some practical line of work, but must be the kind of problem that actually occurs and must be solved by those engaged in that work.

The following problems taken from text books in common use today have to do with practical situations but are not the kind of problems that actually occur in life.

1. A dealer bought 50 gross of buttons for 25%, 10%, 5% off and sold them for $35.91 making a profit of 12%. What was the list price of the buttons per gross?

2. Mr. Day's city tax, at the rate of 13 mills on the dollar, is $84.50. What is the estimated value of his property, if it is assessed at ⅘ of its value?

3. A merchant sold silk at 45 cents a yard above cost, and gained 20%. What was selling price per yard?

4. A man rented a field to a tenant in return for 33⅓% of the grain to be raised. The owner of the field sold his share of the grain for 80¢ a bushel, receiving $240. How many bushels did he have, and how many bushels did the tenant have?

Data. Not only should the problems be of types that actually occur but the data involved in the problems should be true to actual conditions. A problem about butter at 25¢ a pound or shoes at $2.50 a pair is ridiculous and misleading when butter is selling on the local market at 68¢

and shoes at not less than $5. In using text book problems of this kind the teacher and pupils should change the data given to conform to local market conditions.

Problems With Extraneous Data. As has been previously stated, one way in which the school problem differs from the actual life situation is in the fact that the school problem gives the pupils all of the necessary data, whereas in the actual life situation they must first obtain their own data. Further, the text-book problem usually gives only the data necessary for its solution, whereas in the life situation the pupils must often choose from a lot of things that they know or can find out about the situation, those facts or items of data that are necessary for their particular purpose. The school-room problem can be made more nearly like the actual life situation by introducing data not necessary for its solution. For this reason, problems with unnecessary or extraneous data should occasionally be given. Such problems make it necessary for the pupils to exercise their own judgment in selecting, from all of the data given, that necessary to the solution of the problem.

Statement. If the problem is to be interesting and to seem worth while to the pupils it must be stated in such a way as to show how it arises, who has to solve it, and why it needs to be solved. Many of the problems in our texts are absolutely pointless. The following problems are examples.

1. If you have 7¢ how many cents must you get to have 10¢?

2. If you have 8 pieces of candy and divide them into two equal piles how many pieces will there be in each pile?

3. Find the marked price of a chair if the net price is $17 and there is a discount of 5% of the marked price.

These problems as stated are dead and uninteresting and apparently have no reason for existing. Properly stated, however, they become real problems.

1. If you have 7¢ in your bank and want to buy a ball that costs 10¢, how many more cents must you save?

2. If you have 8 pieces of candy and share them equally with your sister, how many pieces will each of you have?

3. If a retail dealer allows a 5% discount from the marked price for cash, what will he have to mark a chair in order to receive a cash price of $17?

Method of Solution. Finally the problems in arithmetic should be solved in the way they are solved in life. If business men figure profits as a per cent of the sales, the school should not figure them as a per cent of the cost; if the paper hanger uses one method of determining the amount of paper needed for a room the pupils in school should not use another method.

Mental Training. Some people argue that the pupils get just as much training from solving artificial problems in artificial ways as they do from solving life problems as they are solved in life. The argument should be turned around: they get just as much training from solving life problems, and besides are having experience with the kind of problems that they are going to meet in life, and are not only developing power to work such problems but are at the same time getting an insight into actual life conditions. The boy who solves problems in school of the type "a merchant bought an article for $10 and sold it for $15. Find the per cent of profit" gets just as much training perhaps as the boy who solves problems of the type "A furniture dealer's cost of doing business is 20% of his sales. At what must he sell a table that costs him $20 in order to make a net profit of 10% of the selling price?" but the first boy is apt to leave school with the impression that a merchant first buys an article, sells it, and then stops to figure profits; while the second boy learns that a merchant must do his figuring before he sells the article and often before he buys it. Preparation for life demands that we

give our pupils experience with and insight into the most important and common life situations involving simple arithmetic. If we do this we will have no time to spend on types of problems that exist nowhere except in our text books.

The Teacher. Proper teaching of problems is impossible unless the teacher looks upon knowledge of the common activities of life as just as important as knowledge of subject matter, methods and books. Sociology—real sociology, not book sociology—is a more important part of a teacher's equipment than psychology. Until the teacher has at least an elementary knowledge of the life situations upon which the school situations are based, she can not do good work and may do positive harm. The best way to get this knowledge is directly from the men who are actively engaged in the various activities of life. The best way to find out what kinds of problems the farmer meets and how he solves them is to go to the leading farmer of the community, and the same applies to the banker, the business man, the carpenter, the plasterer, the paper-hanger, etc.

Problems Must Be Varied

Pupils solve problems in school in order to be able to solve other and similar problems that they may meet in life. The ability to successfully meet any kind of a simple arithmetical situation can not be developed by meeting a few types of situations but only by meeting many different kinds. A boy who enters a large automobile factory and does one thing day after day learns to do that one thing remarkably well, but after several years of such experience he is no nearer being a skilled automobile mechanic than he was in the beginning. On the other hand a boy who enters a garage that does a large general repair business meets a new problem every day, and, while he will never

attain the mechanical precision of the first boy at any one thing, he will eventually become a skilled automobile mechanic and be able to successfully cope with almost any repair job that he may encounter. So, in arithmetic, the pupil who meets only a few types of problems may develop a mechanical skill in solving those types but it is the pupil who meets many and varied types, thinking each through as it occurs, that develops real power to apply his arithmetic to all kinds of situations.

Most of our pupils will probably have to solve problems in simple interest some time in their lives, but most of them will never have to solve problems in lumber measure, plastering and paper hanging. It is necessary that the pupils solve problems in interest, but it is not at all necessary that they solve problems in lumber measure, plastering and paper hanging. It is necessary, however, that they be given opportunities to use their knowledge of mensuration in as great a variety of situations as possible, and lumber measure, plastering and paper hanging are at least real, typical applications of mensuration, possessing the further advantage that they can easily be made concrete to the pupils; and, as such, they should be used, unless a sufficient number of more important applications can be found. Thus, in order to have a sufficient variety of real applications that can be made concrete to the pupils it may sometimes be necessary to use situations a knowledge of which is not absolutely essential to success in life. The teacher is never justified, however, in using situations that are not real life situations or that can not be made concrete.

CHAPTER III

TEACHING PUPILS TO SOLVE PROBLEMS

The Steps in Solving a Problem

The Steps. In order that the teacher may be able to properly guide and assist her pupils it is necessary that she clearly understand the thought steps involved in solving a problem in arithmetic. The complete solution of any problem involves four steps:

1. *Getting a clear understanding of the conditions of the problem.* The first step in meeting any problematic situation is to get a clear understanding of the elements of that situation. In the case of arithmetical problems the pupils must have two things clearly in mind: (1) *The Problem;* what they want to find, and (2) *The Data;* the various items of information given in the problem; or that are known or can be found out about the given situation.

2. *Planning the solution.* Having the conditions of the problem clearly in mind, the next step is to plan the method of solution, *i. e.,* the steps that must be taken to get the desired information from the known.

3. *Executing the Plan,* and

4. *Checking the result obtained.* The solution of an arithmetical problem is not complete until the pupils have determined whether or not their answers are correct.

The teacher's duty with respect to these steps is twofold. In the first place she must see that the pupils form correct habits in thinking through problems, that is, she must make these thought steps habitual. In the second place she must systematically develop in the pupils the ability to perform these steps for themselves.

Making the Thought Steps Habitual

If left to their own devices the pupils' tendency is to start doing something before they really know what the problem is or have any definite plan for its solution. They jump blindly into the middle, trusting to luck that they will come out safely at the other end. They often use a "cut and try" or "trial and error" method, first trying one thing and then another until they get the proper result. This is particularly true if answers are given in the book. The author recently watched a sixth grade pupil prepare her arithmetic lesson. She first read the problem hastily, then turned to the back of the book and looked at the answer and by comparing the answer with the numbers given in the problem decided what to do with these numbers in order to get the desired answer. She was quite skilful at this and usually "got the answer" after one, or at the most two or three trials. This girl had not formed proper habits of thought in problem solving but instead had established habits that were extremely harmful and that made success in problem work impossible. From the primary grades up, the teacher should make it just as much her business to establish proper habits in problem solving as in addition, subtraction or any of the fundamentals. In trying to establish habits of proper thinking in solving problems the teacher must remember that the two most important factors in forming any habit are (1) Repetition of the desired method of procedure, and (2) Permitting no exceptions to the desired procedure to occur until the habit is well established.

Repetition. Until the proper habits of thought are well established all problem solving should be done in class under the direct supervision of the teacher and should follow these steps. Suppose the following problem is one of several being solved in class. "If oranges cost 5¢

apiece, how much will you have to pay for 3?" The solution should always start with definite statements of *The Problem* and *The Data.* In stating the data each item of information known should be stated separately and the statements should be first in words rather than in numbers. After reading the problem the first question should be "What are we to find?" When this is stated it should be written on the blackboard, as:

Problem—To find how much we must pay for the oranges.

The next question raised should be "What does the problem tell you?" At first the pupils will probably answer in numbers, as 5¢ and 3. They must be made to see, however, that in thinking through the problem and planning the method of solution, the actual numbers are not important, the essential thing is what these numbers represent, so the important thing to know is that the problem tells us two things, (1) the cost of one orange, and (2) the number of oranges bought. In order to emphasize these two items of information different pupils may be asked to each state *one* thing that the problem tells, and as each is stated it should be written on the board below the statement of the problem, first in words and then in numbers, as:

Data—Cost of 1 orange =5¢.
Number of oranges bought=3.

Then, *and not until then,* the question of the method to be used in solving the problem should be raised. In this problem all the pupils need to decide is whether to add, subtract, multiply or divide. In more difficult problems later on, they should plan out the whole solution before carrying out any of the calculations.

Having decided that they must multiply in the above

problem the pupils would get the answer mentally. In more difficult problems, having determined upon the method of solution, the pupils would next carry out the calculations and finally consider how they could determine whether the answer obtained is correct.

Permit No Exceptions. Until the proper habits are established all of the problem work should be done in class and should follow these steps. Later, in solving simple problems the pupils will grasp the essentials of the problem and plan the solution at sight and almost unconsciously, and after the proper habits of thinking are firmly established they should be encouraged to do this whenever possible, but until that time the teacher should permit no exceptions and should insist that the pupils follow these steps consciously and in the proper order whenever they solve a problem. Even after the proper habits have apparently been established and the pupils start working independently, exercises similar to the above, in which the pupils solve problems in class, must be held frequently in order that the teacher may make sure that the correct method of procedure is being used and can detect and break any bad habits before they become fixed.

Besides making the proper method of procedure habitual, the teacher has to develop the ability to perform these steps successfully. If the teacher goes about this systematically and intelligently from the third grade on, the majority of the pupils by the time they reach the eighth grade will have formed proper habits and have developed the ability to solve any simple arithmetical problem.

Developing the Ability to Grasp the Essential Conditions of the Problem

Language Difficulties. If the problem is met in words the pupils may have difficulty in determining what the

Problem is and what Data is given simply because they do not understand the language of the statement. The statement may be involved or ambiguous, or may involve words that are not in the pupils' vocabulary. In making problems herself the teacher should be very careful that the statement is concise and direct, can be interpreted in only one way, and is in the pupils' own vocabulary. In the case of text book problems, the teacher must clear up all difficulties arising from the language of the problem before the pupils are asked to solve it.

Problems Must be Concrete. Again the pupils may be unable to pick out the essentials of the problem if the situation on which it is based is not clearly understood. As already stated the pupils should never try to solve a problem that is not concrete to them and it is the business of the teacher to see that the problems are concrete or are made so in some way.

Practice. Granting that no language difficulties exist and that the pupils understand the life situation on which the problem is based, they may still have trouble in taking the connected, detailed statement of the problem and abstracting from it its essential elements. There are several things the teacher can do to help develop this ability. The most important of these is plenty of practice. Of course the pupils get practice on this step every time they solve a problem, but this alone is not enough. As long as the pupils are having trouble with this step, frequent drill should be given on this step alone. One of the best methods of doing this is to take a list of problems, have a pupil read the first one, then call on another pupil to state The Problem, and on other pupils to state the various items of Data. When the Problem and the Data have been completely stated in this way for the first problem, go on to the others and handle them in the same way. Concentrating on the first step only without planning the solution

or solving the problem has several advantages. In the first place it saves time and provides more practice on this step in a given time. In the second place it concentrates the pupils' interest and attention on this one step. If permitted to go ahead and solve the problem the pupils' chief interest will be centered on "getting the answer" and they will neglect the important preliminary steps.

Stating Problems to Fit Given Conditions. In solving problems the pupils must be able to read a problem that is stated in detailed, connected language and pick out the essential elements, namely, the Problem and the Data. It will help them to do this if they are practiced on the reverse process, if they are given the Problem and Data, and required to build up around these the detailed statement. For example, the pupils might be asked to state a problem based on the following:

Problem—To find the Net Price.
Data— Marked Price =
% of Discount=

The first problems the pupils state will probably not be real problems at all but only a statement of the given data in connected form such as, "The Marked Price of a certain article is $25 with a discount of 10%. Find the Net Price." With a little practice and direction, however, they will soon be able to state real, live, interesting problems, as "A. B. Smith and Co. is holding a clearance sale and is giving a 20% discount on all boys' suits. What would I have to pay for a suit if it is marked $12?"

Sometimes actual figures may be given in stating the Problem and Data to the pupils but usually the pupils should supply the figures for themselves as this serves to make them realize that the important thing is not the figures themselves but what they stand for. The pupils should criticize each others' problems, considering

whether they are well stated, whether they are real life problems, and whether the data given is in accordance with actual life conditions. Plenty of practice of this kind in making problems to fit given sets of conditions will not only help the pupils greatly in determining the essential conditions in a given problem but will also help to develop an insight into actual life conditions.

Developing the Ability to Plan the Solution

Practice. Although pupils sometimes have difficulty with the first step it is the second, or the planning of the solution that is the most difficult and gives the greatest trouble. Plenty of practice must be given on this step. Much of this practice will be obtained in connection with the regular work in solving problems, but this needs to be supplemented by extra drill on this step, which is not only the most difficult but also the crucial step in solving any problem. Again, a good way to do this is for the teacher to take a list of problems and have the problem read, call on different pupils to state the Problem and the various items of the Data and then have the pupils plan the solution, without carrying out the solution after it is planned. A class exercise of this kind concentrates the pupils' attention on this important step and does not permit it to be distracted by "getting the answer." Oral exercises in planning the method of solution should be given almost daily. The problems used should involve the different processes and usually the numbers involved should be simple so that the pupils' minds may be left free for the thought work. The statement of the method of solution should be in terms of words rather than in terms of figures and the pupils must be led to form the habit of clear, concise, and definite expression. This work should be started as early as the third grade with simple one-step problems and continued systematically and regu-

larly through the elementary schools with problems of gradually increasing difficulty.

Careful Gradation of Difficulty. In developing the ability to plan the solution careful gradation of the problems is of the greatest importance. In the second and third grades the pupils must develop the ability to solve one-step problems. This means that in such a problem they must be able to determine whether to add, subtract, multiply or divide. In the fourth grade they must develop the ability to plan the solution of two-step and in the upper grades of still more complicated and difficult problems. If the degree of difficulty of the problems is carefully graded and gradually increased, and if the pupils develop the ability to plan problems of a given degree of difficulty before trying more difficult ones the complex problems of the eighth grade should be no more difficult for the eighth grade pupils than the two-step problems of the fourth grade for the fourth grade pupils, and these in turn should be no more difficult than the one-step problems of the second and third grades are to the pupils of those grades. If the increase in the degree of difficulty comes by big steps instead of gradually, or if the increase comes too rapidly the pupils become discouraged and are driven to mechanical methods of solution.

Teach Arithmetic as a Tool. Pupils will always have trouble in planning the solution of problems unless arithmetic is from the very beginning taught as a tool and the arithmetical fact or process connected in the pupils' minds with the kind of concrete situation in which it is used. Teachers realize that it is their duty to teach their pupils to add, subtract, multiply and divide, but many teachers do not seem to realize that it is just as much their duty to see to it that the pupils become familiar with the various types of concrete situations that may arise involving these processes so that they will recognize these situations.

If the connection is to be formed in the pupils' minds between the facts and processes on the one hand and the concrete situations calling for the facts and processes on the other the two must be presented together and always kept together. This means that as soon as a new fact, series of facts, or process is presented, concrete problems must be given involving the facts or process and that from that time on the abstract drill on the facts or process and the applied problems must be kept side by side, usually in the same recitation. If possible, the need for the new information should come first, but where this is not possible at least the use of the information should follow its acquisition immediately. In the work on the measurement of a circle the pupils might encounter the problem of determining the circumference of a circular plot of ground. They might find it rather difficult to measure the circumference directly, but comparatively easy to measure the diameter. This might lead to the question, "Is there any relation between the circumference and diameter of a circle so that we can find the circumference by measuring the diameter?" Then by measuring the circumferences and diameters of various circular objects the pupils could discover the relation $C=\pi d$ and apply it to the original problem and to other similar problems and to problems where the diameter is found by measuring the circumference.

If the discovery of the new relationship does not arise from the needs of a concrete problem, as soon as the new fact is discovered its uses should be immediately considered. As soon as the pupils discover that $C=\pi d$, they should consider how this fact can be used, namely, that it will enable them to find the circumference when the diameter is known or can be found; and to find the diameter when the circumference is known or can be found. The pupils should then at once use this fact in problems of these types. A new fact or process should never be

taught merely as a fact or process but always as a tool to be used in solving certain types of concrete situations. It has already been stated that less abstract drill is necessary where more time is given to applied problems and where the abstract and applied phases are kept as closely connected as possible. Not only is this true but this close correlation of the abstract and applied is absolutely necessary if the pupils are ever to attain any facility in applying the tools of arithmetic to concrete situations. The old type of organization under which the pupils spent the first three years in acquiring the tools of arithmetic, the abstract facts and processes, and in the fourth year started to apply these to concrete situations, is foredoomed to failure. The only way that the pupils will ever learn to apply their arithmetic is to apply it from the very beginning.

The teacher must make certain that the pupils meet all of the different types of problems that are solved by use of the given facts or process. Thus, there are three types of problems in subtraction with which the pupils must become familiar as illustrated by the following problems:

(*a*) If Mary has to walk 5 blocks to school and Johnny has to walk 8, how many more blocks does Johnny walk than Mary?

(*b*) If you had 10¢ this morning and spent 4¢ for candy, how many cents would you have left?

(*c*) If you have 3¢ and you want to buy a ball that costs 5¢, how many more cents must you save?

These might be called the Comparison or Difference, the Take Away or Remainder, and the Additive Types, respectively. The pupils will not meet problems of all of these types at once, but by the time they complete the third grade they should have had sufficient experience with all three types to recognize them as subtraction problems. There are two types of problems in division which the pupils must learn to recognize.

(*a*) *Measuring Type:* If Mary has 8¢ to spend for candy, how many all-day suckers can she buy if they cost 2¢ apiece?

(*b*) *Partitive Type:* If you have 9 apples and share them equally with Helen and Mary, how many apples will each of you have?

Problems Without Numbers. At first the pupils get an idea of the uses of the facts and processes unconsciously by using them frequently in concrete situations. Thus, having learned some subtraction facts the pupils at once start to use them in concrete problems, and by so doing gradually come to recognize subtraction situations as such and to distinguish them from addition, multiplication and division situations. Later, the pupils should be helped to generalize their ideas of addition, subtraction, multiplication and division situations. One of the best ways of doing this is by means of generalized problems or problems without numbers. The pupils first meet individual problems such as (*a*) If you go to the store and pay 5¢ for a tablet and 3¢ for a pencil, how much must you give the clerk? and (*b*) If oranges cost 5¢ apiece, how much must you pay for 4? Later, they should meet these same types of problems in a generalized form, as:

(*a*) If you buy two articles and know the cost of each, how do you find the cost of both? and

(*b*) If you know the cost of one article and want to find the cost of several of the same kind, what do you do?

All of the important types of problems in arithmetic, after having first been frequently met as individual problems, should be generalized in this way. This is one of the best ways that has ever been discovered to bridge the gap between the ability to solve certain problems of a given type and the ability to solve all problems of that type. Used occasionally, such problems will do little good, but used systematically from the third grade up, they will do

much to overcome the pupils' difficulty in determining "what to do" in a given problem.

The pupils find the work with these problems intensely interesting. A third grade teacher once described to the author her experience in using such problems. The first day the pupils found them rather difficult and, finally, one boy after struggling with one for several minutes gave it up in disgust saying, "Aw, I could do it if it had some numbers in it." A few weeks later, however, the same class thought these problems great fun, so much so that one girl broke out at the end of the recitation with, "Oh, Miss A———, let's have these every day, they are just like conundrums."

To make clear the exact nature of these generalized problems or problems without numbers, a miscellaneous list follows:

1. If you know how many eggs you found this morning and how many this evening, how would you find out how many you found altogether?

2. If you know how much money you handed the clerk and how much you spent, what would you do to find how much change you should get?

3. If you know the amount of goods needed to make one doll's dress, how could you find the amount needed to make three?

4. If you know the cost of one dozen, how would you find the cost of six oranges?

5. If you know the weight of each member of your class, how do you find the average weight?

6. What measurements must you take and what must you do to find the area of a rectangle?

7. What measurements must you take and what must you do to find the number of square feet of plastering needed to cover the walls and ceiling of this room?

8. If you know the circumference of a wheel, how do

you find how many times the wheel will turn in going a mile?

9. If you know the amount paid for a certain property and the rate of commission, how do you find the commission?

10. If you know the time that a sum of money is on interest, the rate, and the sum of money, how do you find the interest?

11. If you know what a dealer paid for some goods, and his selling price, how do you find the per cent. of profit or loss?

12. If you know the present population of your town and the population ten years ago, how would you find the per cent. of increase or decrease?

13. What would you have to know and what would you do to find the per cent. of attendance in your room for the last month?

14. If a merchant knows the list price of a certain article and the per cent of discount, how does he find the net price or what he must pay for the article?

15. What measurements would you take and what would you do to find the number of gallons of water a cistern will hold?

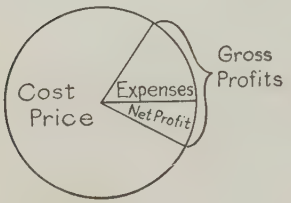
Have Pupils Make Problems to be Solved in a Given Way. Another effective method of developing the ability to plan the solution of problems is to have the pupils make up problems to be solved in a certain way. Thus, in the second and third grades the pupils should be asked to make up problems to be solved by addition, or by subtraction, or multiplication, or division. Making up problems to be solved by addition forces the pupils to keep their eyes open and to notice carefully in what kind of problems addition is used and will help them to recognize addition problems when they meet them. In the fourth grade the pupils should make up problems to be worked by two

processes, say addition and then subtraction. In the same way, pupils can make problems to be worked by finding a fractional part of a number, by finding what per cent. one number is of another, etc.

Clear Grasp of Relations Involved. Planning the method of solution involves bridging the gap from the known to the unknown. The pupils must first have clearly in mind what they want to find out (the unknown) and the data they have at their disposal (the known). They must then plan how to use the known in order to get the unknown. To do this they must have a clear grasp of the relationships between the various quantities in the problem—between the unknown and the known. Anything the teacher can do, or make the pupils form the habit of doing, that will make these relations clear will help the pupils in planning the solution. In the first place if the pupils are to grasp the relations the problem must be concrete to them. The teacher should encourage the pupils to dramatize the situation, to imagine themselves in the place and circumstances of the person who must solve the problem and try to visualize and image the whole situation in all its relations. Most pupils in thinking through problems think in abstract terms and phrases. It is the duty of the teacher to go beyond the words in which the pupils clothe their thoughts and to discover if there are any ideas and images behind these words. If the pupils are to develop the ability to solve problems they must form the habit of thinking in terms of the actual, concrete situations on which the problems are based.

Diagrams and Graphs. Often a diagram or graph will help make clear the conditions of the problem and the relations involved. Thus, in problems in figuring profits and determining selling prices where everything is figured as a per cent. of the selling price, the solution of all of the problems depends upon a clear grasp of the relations

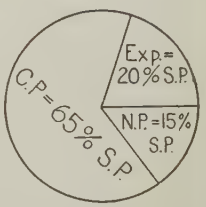
between the Selling Price, Cost Price, Expenses, Net Profit and Gross Profit. These relations can be made clear by means of a diagram such as the one below, where the whole circle represents the selling price of an article.



In working the following problem if the pupils draw the diagram and place directly on it the given data it helps them to see the relation between the known and the unknown.

Problem—If a retail carpet dealer pays \$40 for a rug, and his expenses are 20% of his sales, what must he sell the rug for in order to make a net profit of 15% of the selling price?

From the diagram the pupils see that the expenses and net profit amount to 35% of the selling price, so that 65% of the selling price must pay for the article or equal the cost price of \$40.



Substitute Easy Numbers for Those Given in Problem. Another method of bringing out clearly the relations involved in a problem is to think the problem through, substituting easy numbers for the more complicated ones given. This enables the pupils to think in specific figures instead of general terms and at the same time by using easy numbers avoids distracting their minds with difficult and long calculations.

Methods of Checking the Solution of Problems

1. *Check by solving another problem made from the original problem by interchanging the quantity found and one item of the given data.* Suppose the original problem is, "A publisher wishes to receive a net price of \$1.20

for a book. At what must he list it if he allows a 20% discount from the list price?"

Problem—To find the List Price.
Data— Net Price=$1.20.
Discount=20% List Price.

Solution—80% List Price=$1.20.

$$\text{List Price}=\$1.20\times\frac{5}{4}=\$1.50.$$

This result could be checked by taking the list price found and deducting the 20% discount to see if it gives the proper net price of $1.20.

Check—

	$1.50
Less 20%	.30
	$1.20

"A man bought a house and lot for $9500. The insurance, taxes, repairs and other expenses average $246 a year. At what price per month must he rent it to make a profit at 6% each year on his original investment?"

Problem—To find the monthy rent.
Data— Amount of investment=$9500.
Desired return=6% of investment.
Expenses=$246.

Solution—

$9500	
.06	
$570.00	yearly return
$246.00	expenses
12)$816.00	yearly rent
$68.00	monthly rent

This result could be checked by taking the monthly rent found ($68.00) and figuring the rate or return it will give to see if it really will give the desired return of 6%.

Check—

\$68
12
136
68
\$816
\$246
\$570

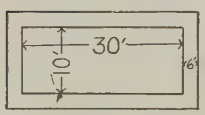
.06 or 6%
9500)\$570.00
570.00

2. *Check by solving the problem by another method.* Often a problem may be solved by several different methods. Where this is the case an effective method of checking is to solve it in two different ways and compare the results. The following problem illustrates this method of checking.

A rectangular grass plot 10 ft. by 30 ft. is to be surrounded by a cement walk 6 ft. wide. Find the number of square feet of cement in the walk.

Problem—To find the number of square feet in walk.

Data— See figure.



Solution—

42
6
252
2
504

6
10
60
2
120
504
624 sq. ft.

This result could be checked by first finding the area of the whole rectangle and the area of the grass plot in the center and subtracting.

Check—

30
10
300

42
22
84
84
924
300
624 sq. ft.

3. *Check by going over the work a second time.* If no better method of checking the result can be found the work should be reviewed carefully in order to detect errors. This is not a very satisfactory method of checking, as the mind in going over the same work in the same way tends to repeat itself, so any error made the first time is apt to be repeated. This method should be used only as a last resort when no more effective way of checking can be found. It should be remembered that errors in the solution of concrete problems may be of two kinds: (1) errors in planning the solution, or in thinking through the problem; and (2) errors in carrying out the solution, or in the mechanical calculations. These calculations should be checked as soon as performed, so in checking the final result the reasoning involved in planning the solution should be gone over carefully to make sure that no error has been made.

Approximate Methods of Checking. Sometimes approximate methods of checking are sufficiently accurate to detect large mistakes. The two most useful approximate methods are: (1) Before carrying out the complete solution, form an approximate estimate of the answer by substituting the nearest easy numbers for the numbers of the problem and use this estimate as a check on the solution when obtained; (2) Put the answer obtained back in its concrete setting to see if it is reasonable.

Estimating Result. The first method may be illustrated by means of the following problem. "A contractor agreed to build a circular basin for a fountain at the rate of 65¢ for each square foot in the bottom of the basin. Find the cost of a basin 38 ft. in diameter." The result can be approximated by calling the radius 20 ft., using $\pi=3$, and calling sixty-five cents ⅔ of a dollar. Then: $20^2=400$. $3\times400=1200$. ⅔ of 1200=$800. This can be used as a rough check on the exact result (which is $737.18) when

obtained, and is sufficiently accurate to detect any large error, particularly the common error of placing the decimal point in the wrong place and getting answers such as \$73.72 or \$7371.80 instead of \$737.18.

Place Result Back in Its Concrete Setting. The second method of checking approximately may be illustrated by the following problems. "The President of the United States receives a yearly salary of \$75,000. Counting 300 working days in the year, find his daily salary." This problem was recently given to several eighth grade classes and they obtained answers ranging all the way from \$2.50 through \$25 and \$2500 to \$25,000. Many of these errors would have been detected if the pupils after getting the result had stopped to think what the result stood for, had put it back in its concrete setting to see if it was reasonable. If they had done this it would have been obvious that \$2.50 was entirely too small for the President's daily pay and that \$25,000 was entirely too large, as his yearly salary is only \$75,000.

Another illustration of the same thing occurred recently in one of the author's normal college classes. On a written test a problem in commercial discount was given in which the students had to figure the list price. One student obtained a result smaller than the net price and never noticed anything wrong. Nothing will serve to detect large errors more effectively than the habit of considering the result in its concrete setting to see if it is reasonable, or entirely out of reason.

CHAPTER IV

FORM OF WRITTEN SOLUTIONS—ANALYSIS

Reduce Written Calculation to a Minimum. In life the important thing in solving problems is to get the correct result with a minimum of figuring. In school, too often, instead of developing this ability and habit we insist on long, complicated forms of solution. In most of the work with problems in the grades the pupils should be required to get their results with a minimum of figuring, doing as much of the work as possible mentally and putting down the written calculations in the most economical form. The calculations should be carried out with abstract numbers and only the final result "labeled" unless the pupils find it necessary to label other numbers to aid them in their thinking. The pupils should be encouraged to see which can get the shortest solution and should be led to take as much interest and pride in this as in getting the correct result.

As an illustration, take the following problem. "One year a field produced 56,800 pounds of wheat. The next year with better preparation of the soil and greater care in selecting seed, the same field produced 62,500 pounds. How much was the increase worth at 85¢ a bushel?"

Solution—

62500
56800
60)5700
95
.85
475
760
$80.75

Full Written Solutions. The preceding should not be taken to mean that pupils should never be required to use a complete form of solution. On the other hand, a complete form of solution will often aid the pupils in solving difficult problems, so the pupils should be made familiar with good complete forms to use when necessary and should have considerable practice in writing such complete solutions. But as stated before, most of the time the emphasis should be on getting the result with a minimum of written work rather than on the complete solution.

There are three essentials to any good, complete solution. (1) The form of the written solution must follow and emphasize the thought steps taken in solving the problem. (2) The solution must be intelligible. Anyone who sees the solution should be able to get from it, without any further source of information, the conditions of the problem and the method used in solving it as well as the result obtained. (3) The solution must be as brief as the other two requirements will permit. Any form that satisfies these conditions is good, and any form that fails to satisfy all three of these conditions is bad.

The Written Solution Must Follow and Emphasize the Thought Steps

The thought steps in solving any problem as already stated are: (1) Getting a clear understanding of the Problem and the Data, (2) Planning the solution, (3) Executing the plan, and (4) Checking the result obtained. The complete written solution should emphasize and follow these steps if it is to aid the pupils in solving difficult problems. It follows that the large subdivisions of the solution will be *Problem, Data, Plan, Solution,* and *Check.* These or similar headings should be used to set off the different parts of the complete solution.

Problem and Data. In stating the Problem and Data, the statements used should be as brief and as simple and direct as possible. The various items of the Data should be stated separately, and in order to aid the pupils in their thinking they should be stated first in words. In the problem used above as an illustration the solution would start with some such statements as:

Problem—To find the value of the increase.
Data— No. lbs. wheat produced first year=56,800.
No. lbs. wheat produced second year=62,500.
Value per bushel=85¢.

Some teachers prefer to write the Data first and the Problem second. There are arguments in favor of each order. In life when we meet an actual situation the problem usually arises first and the data necessary to solve it is then obtained. Thus, if for some reason we need to know the number of acres in a field, the need for this knowledge, the Problem, first arises, and in order to solve this problem we then obtain the Data by measuring the length and width of the field. Another argument for placing the Problem first is that this conforms to the method of procedure in other subjects where the problem method of teaching is used. About the only arguments for the other order are that it is the traditional order and it is the order usually used in higher mathematics.

The Plan. Some teachers have the pupils include this step in the written solution. Thus, in the problem above the plan might be written as follows:

Plan—1. Find number of lbs. of increase by subtracting.
2. Change the increase from lbs. to bushels by dividing by 60, the number of pounds in a bushel.
3. Find the value of the increase by multiplying by $0.85, the value of one bushel.

Sometimes it is possible to indicate the processes to be performed in a single statement before performing any of the calculations. This may be done in the above problem.

Plan—

$$\text{Value of increase} = \frac{(62{,}500 - 56{,}800)}{60} \times .85.$$

In building up and explaining this statement the pupils should be able to state in words what the result of each step will be. The first step, 62,500—56,800, will give the number of pounds increase, this result divided by 60 will give the number of bushels increase, and this result multipled by .85 will give the value of the increase in dollars and cents.

Many teachers object to writing out the plan of the solution on the grounds that it takes too much time. It is not necessary to make this step a written one, providing the teacher makes sure that the pupils formulate a definite plan before attempting the solution. The formulation of the plan may be oral instead of written if the teacher prefers. It must be remembered, however, that full written solutions ought not to be demanded all of the time but only when needed to help the pupils in their solution. The time taken to write out the plan, then, is not a serious objection, and many teachers find a written plan one of the most effective methods of making certain that the pupils have a definite plan in mind before they start adding, subtracting, etc.

The Solution. To be intelligible, the solution must indicate, in order, all of the steps taken in solving the problem. For the sake of brevity, however, it is usually better simply to indicate the calculations and the results, performing the actual calculation on scratch paper. This is particularly true in the case of long multiplications and divisions.

In indicating the solution two forms are in common use,

the horizontal and the vertical. Sometimes one of these is the shorter and sometimes the other. In many cases the two forms can be combined to advantage. The solution of the problem used above is given below first in the horizontal and then in the vertical form.

Solution—No. lbs. increase=62,500—56,800=5700.

$$\text{No. bushels increase}=\frac{5700}{60}=95.$$

Value of increase=95×\$.85=\$80.75.

Solution—

62500
56800
60)5700
95
.85
475
760
\$80.75 value of the increase.

In the horizontal solution the result of each step is labeled and the label is put first instead of last. This, again, forces the pupils to plan ahead, to think first instead of last. Thus, in the first step the pupils must think "I must first find the number of pounds increase by subtracting" and then subtract. If the label is written last it encourages them in their natural tendency to do something first and then try to find out what the result obtained means.

Check. Every problem solved should be checked in some way. The check may be either written or oral. If the second of the two approximate methods of checking suggested, or the method of reviewing the reasoning is used, the check must be oral. If either of the other methods is used, the check would be written and should be made part of the complete written solution. The easiest way to check the result obtained in the above problem would

be to estimate the result, using approximate figures. The increase is roughly 6000 lbs. or 100 bushels, which at 85¢ a bushel would be worth $85.00. This result, it should be remembered, is evidently slightly too large, as the increase is somewhat less than 6000 pounds. This rough estimate of the result should be made when the solution was planned and before it was carried out, and used as a rough check on the result finally obtained, namely, $80.75.

A satisfactory complete solution of the above problem, gathering together the various parts already worked out, would be as follows:

Problem—To find the value of the increase.

Data— No. lbs. wheat produced first year=56,800.
No. lbs. wheat produced second year=62,500.
Value per bushel=85¢.

Plan— Value of increase=$\frac{62{,}500-56{,}800}{60}\times.85$.
Estimated result=$85 (known to be too large).

Solution—No. lbs. increase=62,500−56,800=5700.
No. bu. increase=$\frac{5700}{60}$=95.
Value of increase=95×$.85=$80.75.

Check— Estimated result=$85 (known to be too large).
Result obtained=$80.75.

The Analysis of Problems

Definition and Purpose. An analysis of a problem in arithmetic is an orderly, concise statement of the thought steps that lead to the discovery of the solution. Every time the pupils solve a problem they must analyze it. Usually they do this thinking to themselves, and the teacher sees only the result of the thinking—the solution of the problem. Sometimes, however, the pupils must be required to state the thought steps by which they discovered their

solutions. Such statements given by the pupils give the teacher valuable information, as they expose the pupils' thinking and enable the teacher to locate and overcome difficulties and to make certain that the pupils are forming habits of thought that will aid them in solving more difficult problems.

In the case of problems solved outside of class the teacher obtains this information by means of the complete written solution discussed above. Often, however, the pupils should do their thinking in class and aloud. This is particularly important when problems of a new type or of increased difficulty are first met. These should be analyzed orally in class by the pupils before they try to solve them. This makes it possible for the teacher to locate and overcome difficulties and helps to establish proper habits of thought. Preliminary class analysis is necessary at this time but must be dropped as soon as possible; if continued too long it will cripple the pupils. Eventually they must be able to analyze and solve the problems without assistance, but the best way to bring this about is to anticipate difficulties and form proper habits, in the beginning.

If the pupils have solved a list of assigned problems successfully there is nothing gained by having them explain or analyze their solutions. If, however, a pupil has tried to solve a problem by himself and has failed the teacher should not be satisfied with merely knowing that he has failed, but should try to find out the cause of his failure. The best way to locate the difficulty is to have the pupil try to think through the problem out loud. Often in this way the teacher can locate and help the pupil overcome the trouble. If the pupil can not overcome his own difficulty with the aid of the other pupils and the teacher, then the next best thing is to have another pupil explain or analyze the problem for his benefit.

The traditional problem recitation, in which the pupils

write the solutions of the problems assigned on the blackboard and then analyze or explain them, is a very effective way of killing time—it serves no other purpose. Such a recitation always reminds the author of a science teacher in a high school who was once forced to take a class in English for a few weeks. The recitation period was forty minutes long but seemed forty years to this teacher, as he could think of nothing to do to occupy the time. The class was reading Ivanhoe, and finally having exhausted all his resources, in sheer desperation he sent the class to the board and kept them there the whole forty minutes copying Ivanhoe word for word from their books on to the board. There is nothing gained by having a pupil analyze or explain a problem that he has successfully solved unless he does it for the benefit of some other pupil. If a list of problems has been assigned for home work, a few minutes at the beginning of the hour should usually be sufficient for the teacher to check up on the work and to clear up difficulties. The rest of the hour should then be spent on working more problems in class rather than on explaining problems already worked.

Character of the Analysis. Since the purpose of the analysis is to expose the pupils' thinking, it follows that it should be in the pupils' own words and should not be formalized. A pupil may recite very glibly a formal analysis such as the following: "If one pencil costs 4¢, three pencils will cost three times as much as one pencil, or 3 times 4¢ which is 12¢," but the chances are that the thinking back of the analysis is no more his own than is the phraseology in which it is expressed. Pupils should be permitted to explain or analyze a problem in their own way and in their own words. It should be remembered, however, that the analysis is simply a statement of the thinking necessary in solving the problem and should therefore make clear the steps involved. It should start with a statement of the

Problem, and the Data, and this should be followed by a statement of the various steps in the solution and the results of each step.

Although the analysis of problems is very important, it can easily be overdone. An explanation or analysis as such is worthless; it is only worth while when it is used to assist the pupils in solving problems. It should always be used with a definite purpose in mind. The only way that pupils will ever develop the ability to solve problems is by solving problems. Much of the time often spent in explaining and talking about the solution of problems had much better be spent in solving more problems.

CHAPTER V

MEASURING THE ABILITY TO USE ARITHMETIC

Tests for measuring the pupils' mastery of the fundamentals of arithmetic have been discussed in a previous chapter. These by themselves give only a partial measure of arithmetical ability. As has been seen, the ability to use the fundamentals in life situations is just as important as a mastery of the fundamentals themselves. The tests already considered, therefore, need to be supplemented by others to measure the ability to apply arithmetic to concrete situations.

Several tests have been devised for this purpose. They are not as satisfactory as the tests already described, but properly used they give the teacher much valuable information. The chief defect of all the following tests is the fact that they measure ability to solve a certain type of more or less artificial text book problem, rather than the ability to apply arithmetic to actual life situations. No one has as yet attempted to devise tests to measure the latter ability.

One of the first tests devised to measure the ability to solve problems was the Stone Reasoning Test,* which is given below.

Stone Reasoning Test

Solve as many of the following problems as you have time for; work them in order as numbered:

1. If you buy 2 tablets at 7 cents each and a book for

*Published by Teachers College, Columbia University. Reprinted by permission of author and publisher.

65 cents, how much change should you receive from a two-dollar bill?

2. John sold 4 Saturday Evening Posts at 5 cents each. He kept ½ the money and with the other ½ he bought Sunday papers at 2 cents each. How many did he buy?

3. If James had 4 times as much money as George, he would have $16. How much money has George?

4. How many pencils can you buy for 50 cents at the rate of 2 for 5 cents?

5. The uniforms for a baseball nine cost $2.50 each. The shoes cost $2 a pair. What was the total cost of uniforms and shoes for the nine?

6. In the schools of a certain city there are 2,200 pupils; ½ are in the primary grades, ¼ in the grammar grades, ⅛ in the High School and the rest in the night school. How many pupils are there in the night school?

7. If 3½ tons of coal cost $21, what will 5½ tons cost?

8. A newsdealer bought some magazines for $1. He sold them for $1.20, gaining 5 cents on each magazine. How many magazines were there?

9. A girl spent ⅛ of her money for car fare, and three times as much for clothes. Half of what she had left was 80 cents. How much money did she have at first?

10. Two girls receive $2.10 for making button-holes. One makes 42, the other 28. How shall they divide the money?

11. Mr. Brown paid ⅛ of the cost of a building; Mr. Johnson paid ½ the cost. Mr. Johnson received $500 more annual rent than Mr. Brown. How much did each receive?

12. A freight train left Albany for New York at 6 o'clock. An express left on the same track at 8 o'clock. It went at the rate of 40 miles an hour. At what time of day will it overtake the freight train if the freight train stops after it has gone 56 miles?

The twelve examples included in the test are not of equal difficulty. Their relative difficulty was determined

by giving them in chance order to one hundred sixth grade pupils. The comparative difficulty or value so determined is:

Problem	Value	Problem	Value
1	1	7	1.2
2	1	8	1.6
3	1	9	2
4	1	10	2
5	1	11	2
6	1.4	12	2

Method of Giving and of Scoring. The pupils are given fifteen minutes in which to solve as many problems as possible. This time was chosen because it was found to be "the time in which a majority of the pupils worked through the first six or seven problems and in which practically no pupil completed the test." In scoring, the problems are marked right or wrong, according as the reasoning is right or wrong, regardless of whether the answer is correct or not. If the problem contains more than one reasoning step credit is given for each step reasoned correctly. Thus, if there are three reasoning steps and only one is reasoned correctly, ⅓ credit is given. If a problem is unfinished, credit is given for the reasoning steps correctly performed. The individual scores are found by adding the values of the problems reasoned correctly. Thus, if a pupil has the first four correct, the fifth two-thirds correct, and in the sixth has correctly performed two of the five reasoning steps, his individual score would be $1+1+1+1+\frac{2}{3}+$ ($\frac{2}{3}$ of 1.4)$=5.23$. Class scores are obtained by adding the individual scores and changing to the basis of 100 pupils. Thus, if the sum of the individual scores of a class of 27 pupils is 157.3, the class score on the basis of 100 pupils would be $\frac{100}{27}\times 157.3=582.59$.

Accuracy. The scores so obtained measure only the

speed of the pupils' reasoning. The accuracy of the reasoning or the per cent of problems reasoned correctly may also be figured. It is found by dividing the number of problems reasoned correctly by the total number attempted. In figuring the accuracy of an individual pupil, fractions should be used, if necessary, in counting both the number correct and the number attempted. In figuring the accuracy of a group of fifty or more pupils, however, it is sufficiently accurate to count only the number of problems entirely correct and to neglect problems started but not completed in counting the number attempted.

Standards. Stone has recently proposed the following tentative standards:

Score to be reached or exceeded by 80% or more of the pupils.

Grade	Speed	Accuracy
5	5.5	75%
6	6.5	80%
7	7.5	80%
8	8.75	90%

The following table gives the *median** scores recently obtained in two school systems:

Grade	Butte, Montana	Salt Lake City, Utah
5	2.2	3.7
6	3.9	6.4
7	5.8	8.6
8	7.7	10.5

The Starch Scale‡

Starch has prepared a scale for measuring ability to solve problems somewhat similar in plan to the Woody Scale for measuring ability in the fundamentals. The scale consists

*For definition of median see page 224.

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of a series of fifteen problems of unequal difficulty. The relative value or difficulty of the problems has been determined experimentally and the problems arranged in order of difficulty, starting with the easiest. A pupil's score is the highest step done correctly. The test follows:

Arithmetical Scale A

(Prepared by D. Starch)

The numbers in parenthesis are the actual scale values of the problems.

Do the following problems in the order given. Do all the work on the back of this sheet.

Step 1 (.4)

Mary had 4 apples and her mother gave her 7 more. How many apples did Mary then have? Answer.

Step 4 (3.8)

Sam had 12 marbles. He found 3 more and then gave 6 to George. How many did Sam have left? Answer.

Step 6 (5.9)

John sold 4 Saturday Evening Posts at 5 cents each. He kept ½ the money and with the other ½ he bought Sunday papers at 2 cents each. How many did he buy? Answer.

Step 7 (6.7)

If you buy 2 tablets at 7 cents each and a book for 65 cents, how much change should you receive from a two dollar bill? Answer.

Step 8 (7.7)

How many pencils can you buy for 50 cents at the rate of 2 for 5 cents? Answer.

STEP 9 (9.2)

A farmer who had already sold 1897 barrels of apples from his orchard hired 59 boys to pick the apples left on his trees. Each boy picked 24 barrels of apples. What was the total number of barrels the farmer got from his orchard that year? Answer.

STEP 10 (10.3)

A newsdealer bought some magazines for $1. He sold them for $1.20, gaining 5 cents on each magazine. How many magazines were there? Answer.

STEP 11 (11.3)

In the schools of a certain city there are 2200 pupils; ½ are in the primary grades, ¼ in the grammar grades, ⅛ in the high school and the rest in the night school. How many pupils are there in the night school? Answer.

STEP 12 (11.7)

If 3 and ½ tons of coal cost $21, what will 5 and ½ tons cost? Answer.

STEP 13 (12.9)

A school in a certain city used 2516 pieces of chalk in 37 school days. Three new rooms were opened, each room holding 50 children, and the school was then found to use 84 sticks of chalk per day. How many more sticks of chalk were used per day than at first? Answer.

STEP 14 (14.2)

A girl spent ⅛ of her mony for car fare, and three times as much for clothes. Half of what she had left was 80 cents. How much money did she have at first.

Answer.

STEP 15 (15.1)

John had $1.20 Monday. He earned 30 cents each day on Tuesday, Wednesday, Thursday and Friday. Saturday

morning he spent one-third of what he had earned in the four days. Saturday afternoon his father gave John half as much as John then had. How much did his father give John? Answer.

Name..... City .....
Grade ..... Date .....
School..... Age..... Years..... Months.....

Standards. The following Standard June Scores are derived from 2515 pupils in eighteen schools:

Grade	3	4	5	6	7	8
Score	4.5	6.2	7.8	9.4	11.0	12.6

The Courtis Tests*

Courtis included two reasoning tests in his Standard Research Tests, Series A. Test No. 6 is a speed reasoning test. It consists of sixteen one-step problems of supposedly equal difficulty. The pupils do not solve the problems but simply write down after each the operation they would use if they were going to solve it. The time for the test is one minute. The pupil's score is the number attempted and the number right. The test follows:

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"Measure the efficiency of the entire school, not the individual ability of the few"

SCORE
No. Attempted....
No. Right .....

Arithmetic—Test No. 6. Speed Test—Reasoning

Name..... *School*..... *Grade*.....

Do not work the following examples. Read each example through, make up your mind what operation you would use if you were going to work it, then write the name of the operation selected in the blank space after the example. Use the following abbreviations: "Add." for addition, "Sub." for subtraction, "Mul." for multiplication. and "Div." for division.

	Operation	
1. A boy played several games of marbles. He had 15 marbles when he began to play and he won 13. How many marbles did he have when the games were over?....		
2. One day in vacation a boy went on a four-day fishing trip. He caught just 12 fish on each of the four days. How many fish did he catch on the trip?.....		
3. In a certain class, the heaviest child weighed 126 pounds. If he was three times as heavy as the lightest child how much did the lightest child weigh?.....		
4. Two sisters practiced lifting weights. One girl could lift 90 pounds, the other girl but 53 pounds. How many pounds more could one girl lift than the other?.....		
5. A girl watching from a window saw 27 automobiles pass a school in an hour. Each auto carried 4 lamps. How many lamps did she see in the hour?.....		
6. A school was very crowded. It was necessary to place 18 new seats in a room in which there were 33 seats. How many seats were there in the room then?.....		
7. The children of a school gave a sleighride party. There were 650 children in the school, and 463 at the party. How many children did not go to the party?.....		
8. A box of colored chalk in which there were 144 pieces lasted just a year. If each teacher used 24 pieces, how many teachers were there in the school?.....		
9. On a playground there were two groups of children. If there were 47 children on the grounds, and 26 in one group, how many children were there in the other?.....		
10. Two girls played a number game. One girl made 58 points, the other 19 points. The score of the winner was how many times the score of the girl that lost?.....		
11. Two boys in a woods gathered nuts, which they put into one pile. One boy found 215 nuts, the other 346 nuts. How many nuts were there in the pile?.....		

	Operation	
12. A girl, making a collection of postal cards, bought 7 packages in one day. Each package contained 12 cards. How many cards did she buy that day?.....		
13. A girl found that it took her 27 minutes to walk from her home to her school which was 18 blocks away. How long did it take her to walk a block?.....		
14. A club of boys sent their treasurer to buy a baseball. They gave him 75¢ and he spent 45¢. How much money did he have to take back to the club?.....		
15. In a school the 7 sections of the eighth grade were each 13 children smaller in June than in September. How many children left the grade during the year?.....		
16. A boy walked 9 blocks from his house towards a school to meet his chum. They walked the remaining 7 blocks together. How far did the boy live from the school?		

Standard Scores.

Grade	Attempts		Rights	
3	2.5	2.1	1.5	1.0
4	3.5	3.4	1.8	1.8
5	4.2	4.2	2.6	2.7
6	4.9	4.8	3.5	3.5
7	5.6	5.5	4.5	4.5
8	6.4	6.3	5.7	5.4

The first score in each case is the standard proposed by Mr. Courtis, based on 66,837 individuals. The second score is the median score in the city of Boston based on 18,259 individuals.

Test No. 8, Series A, consists of eight two-step problems of supposedly equal difficulty. The pupils are given six minutes for the test and are asked to solve the problems. The pupil's score is again the number attempted and the number right. The test follows:

"Measure the efficiency of the entire school, not the individual ability of the few"

SCORE
No. attempted .....
No. right .....

Arithmetic Test No. 8—Reasoning

Name..... *School*..... *Grade*.....

In the blank space below, work as many of the following examples as possible in the time allowed. Work them in order as numbered, entering each answer in the "answer" column before commencing a new example. Do not work on any other paper.

	Answer	
1. A farmer who had already sold 1897 barrels of apples from his orchard hired 59 boys to pick the apples left on his trees. Each boy picked 24 barrels of apples. What was the total number of barrels the farmer got from his orchard that year?.....		
2. At a candy pull, 49 children, 27 girls and 22 boys, made 3 kinds of candy in 90 minutes. The total number of pieces made was 2765, of which 560 were eaten at the party. The rest were shared equally. How many pieces did each one get?.....		
3. On a bicycle trip a party of boys rode 15 miles the first hour, 17 miles the second, 11 miles the third, and 14 miles the fourth, then stopped for the day. If they rode as many miles on each of 27 days, what was the total length of the trip?		

	Answer	
4. A group of 12 children took turns in counting the automobiles that passed a school. They counted 1833 autos in 39 hours. Six months later they counted 58 autos an hour. How many more machines passed the school each hour than at first?		
5. A party of 5 children traveled 12 miles from a school to a woods to gather nuts. One child found 20 nuts, a second 25 nuts, a third 83 nuts, a fourth 140 nuts, and the last 160 nuts. They wanted 600 nuts altogether. How many more did they need?		
6. During the year a room in a school used 9 boxes of chalk, each holding 144 sticks. There were 48 children in the room. If each child had been given his share at the beginning of the year, how many sticks would each have received?		
7. At 2 Christmas entertainments for poor children 2400 presents were given away the first night, and 3000 presents the second night. If 216 rooms in the 22 schools of the city each contributed its share of the presents, how many did one room give?		
8. A girl read 105 pages in both her reader and her history. By counting she found there were 2342 letters on one page of her history and 2295 letters on a page of her reader. How many more letters had she read in one book than in the other?		
TOTAL		

Standard Scores.

Grade	Attempts		Rights	
3	2.5	2.3	0.5	0.5
4	2.9	4.1	0.7	0.5
5	3.1	3.7	1.0	0.7
6	3.4	3.8	1.4	1.1
7	3.7	4.3	1.9	1.7
8	4.0	4.6	2.5	2.4

The first score in each case is the standard proposed by Mr. Courtis, based on 66,837 individuals. The second score is the median score in the city of Boston based on 18,259 individuals.

MONROE STANDARDIZED REASONING TESTS IN ARITHMETIC*

There are three tests of fifteen problems each in this series, the first being intended for grades 4 and 5, the second for grades 6 and 7, and the third for grade 8. The papers are graded both for correct principle and correct answer and separate scores are kept. The problems on a given test are not of equal difficulty but are assigned different values. Thus in Test I in the third problem, 3 points are given for correct principle and 2 for correct answer (P=3, C=2). The time for each test is 25 minutes. The tests follow:

Test I. For Grades 4 and 5

1. Mr. Black received $2 a yd. for broadcloth. He sold 78 yds. How much did he receive? (P=2, C=2.)

2. If a man has $275 in the bank and draws out $70 how much has he left in the bank? (P=2, C=2.)

3. Oats weigh 32 lbs. to a bushel. How many bushels are there in a load weighing 1344 lbs.? (P=3, C=2.)

4. Find the cost of 864 bags of coffee of 130 lbs. each at 6 cents per pound. (P=4, C=3.)

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5. Mary worked 20 examples on Monday and 19 on Tuesday. How many did she work in the two days? (P=2, C=1.)

6. After traveling 75 miles, how far must I go to complete a trip of 95 miles? (P=2, C=1.)

7. A car contains 72,060 lbs. of wheat. How much is it worth at 87 cents a bushel? (P=4, C=3.)

8. At 3 cents per foot, what is the cost of sufficient picture molding to go around a room 14 ft. by 14 ft.? (P=4, C=2.)

9. How many pounds of hay are raised on 6 acres at 3804 pounds to the acre? (P=2, C=2.)

10. A Kansas farmer bought 80 acres of cheap land for $240. Oil being found on his farm, he sold his land for $60,000. What was his profit? (P=3, C=2.)

11. Find the contents of a box 3 ft. long, 2 ft. wide and 2 ft. high. (P=4, C=1.)

12. Three boys buy a rowboat for $15.75, sharing the expense equally. Find how much each boy has to pay. (P=3, C=2.)

13. By selling butter at 24 cents per pound a lady rereceived enough money to buy 48 lbs. of coffee at 20 cents per pound. How many pounds of butter does she sell? (P=4, C=2.)

14. A house rents for $35 a month. This is how much a year? (P=2, C=2.)

15. Find the change from a two-dollar bill in paying the following amounts on packages to be sent by parcel post: 12 cents, 20 cents, 8 cents, 14 cents, 32 cents. (P=3, C=2.)

Test II. For Grades 6 and 7

1. A girl having ¾ yd. of ribbon bought ⅛ yd. more. What part of a yard had she then? (P=2, C=1.)

2. A piece of ribbon 4⅜ yds. long is cut from a bolt containing 10 yds. How many yards are left? (P=2, C=2.)

3. There are 31.5 gallons in a barrel. How many gallons are there in 63 barrels? (P=2, C=2.)

4. If a horse eats ⅜ bu. of oats a day, how long will 6 bu. last? (P=3, C=2.)

5. When a 20-pound cheese is worth $1.90, how much will a 10-pound cheese cost? (P=1, C=1.)

6. Four loads of hay are to be put into a barn. The first load weighs 1.125 tons; the second, 1.75 tons; the third, 1.8 tons; the fourth, 1.9 tons. Find the weight of the four loads. (P=1, C=2.)

7. A baker used ⅗ lb. of flour to a loaf of bread. How many loaves could he make from a barrel (196 lbs.) of flour? (P=3, C=2.)

8. My telephone bill is $12.85 a month. At that rate how much should I pay in 2¾ years? (P=2, C=2.)

9. A man spends $6.50 for board, $12.25 for clothing, $5.20 for books, and had $12 left. How many dollars and cents had he at first? (P=1, C=2.)

10. A boy saves 1¾ cents on a picture by doing his own developing and printing. This makes a saving of how much on each dozen pictures? (P=2, C=1.)

11. Make out the following account for a day. Cash on hand, $174.30; Receipts, mdse., $12.50; $6.75, $0.42, $17.30, $9.50, $42.75; Expenses, Perry and Co., bill, $75.82. (P=3, C=3.)

12. Muslin is to be bought for 12 new curtains each requiring 2⅞ yds. How much muslin cost at 12½ cents a yard? (P=3, C=3.)

13. A market man has 7850 pounds of ice put into his refrigerator at one time. How much does it cost at $3.90 a ton? (P=2, C=3.)

14. A man bought two suits of clothes, one costing $35.75 and the other $28.50. How much more did the one cost than the other? (P=1, C=1.)

15. A farmer raised 500 bushels of wheat on a field of 40 acres. What was the average yield per acre? (P=2, C=2.)

Test III. For Grade 8

1. A cow that cost $56 was sold at a gain of 12½%. What was the gain? (P=2, C=1.)

2. A man invested $1750 and lost $192.50. What per cent. did he lose? (P=2, C=1.)

3. The wages of the men in a certain factory are to be raised 20% from their present scale. How much will a man get after the raise who is getting $2.25 before the raise? (P=2, C=1.)

4. A hardware dealer sold a furnace for $180 at a gain of 5%. What did the furnace cost him? (P=3, C=2.)

5. If a man saves $18.75 out of his salary of $1250, what per cent. does he save? (P=2, C=1.)

6. A boat carried 3125 tons of iron ore. This ore will yield 43.8% of iron. How many tons of iron in the cargo? (P=2, C=3.)

7. What is the tax on property assessed for $12,480 at $13.50 a thousand? (P=2 ,C=2.)

8. If 33⅓% of the weight of meat is lost in shrinkage when cooked, what ought a ham weighing 12 lbs. when raw to weigh when baked? (P=2, C=1.)

9. Find the interest at 5% on $640 for 4 years. (P=2, C=1.)

10. A salesman sold $75,000 worth of goods one year. His commission was 7½% of his sales. What did he earn? (P=2, C=2.)

11. Ellen bought a pocketbook for $0.90 just after Christmas. She paid 40% less than was asked for it before Christmas. Find the price before Christmas. (P=3, C=1.)

12. A fast train runs from Chicago to a station 356.4

miles distant in exactly 9 hours. What is the average rate of the train? (P=1, C=1.)

13. Books that cost $1.50 wholesale were sold at a gain of 10%. What was the selling price? (P=2, C=1.)

14. A dealer bought $167.40 worth of clocks and sold them at a profit of 33⅓%. How much did he gain? (P=2, C=1.)

15. A clerk had his weekly wages increased $3, or 16⅔%. What were his wages before the increase? (P=3, C=1.)

Tentative Standards. These tests have not been in use long enough for standard scores to be available, but the following tentative standards have been proposed by their author, based on the scores made in a preliminary series of tests in which the same problems occurred.

Grade:	4	5	6	7	8
Correct Principle.....	12	20	15	21	18
Correct Answer.....	7	11	9	14	10

APPENDIX

LESSON PLANS

A Development of the Number Idea Six
First Grade*

Subject Matter	*Method of Procedure*
Counting.	What have you been doing with the numbers we have had?
Chickens, rabbits, money, etc.	What have you counted?
One Two Three Four Five 1 2 3 4 5	Look at the numbers on the board. Come and count them. Point to the numbers.
Two Three One Four Five 2 3 1 4 5	See if they look the same in this row.
Several pupils read the numbers.	Who can give them?
We have four pictures on our wall. Here is the word four. Here is the number four.	Count the pictures on our wall and find the word and number.
Two doors, five windows, one clock, and three chairs in the school-room.	Same procedure with each of these as with the four pictures.
Row of six boys and girls.	Here is a big row of boys and girls. Come and count them.
1—2—3—4—5—?	Harry tries and counts to five but is not sure of the next number.
The next number is six.	Who can tell Harry what number comes after five?
Put six after five. One Two Three Four Five Six 1 2 3 4 5 6	Let us see how this new number looks on the board. Where shall I write it?

*Plan by Miss Lucy H. Meacham, Critic Teacher, First Grade, Bowling Green State Normal College, Bowling Green, Ohio.

Here are six red sticks.	Bring six red sticks.
1—2—3—4—5—6. Here is the figure six. Here is the word six.	You may count them. Find the word and number.
Six blocks, girls' hair ribbons, buttons on boy's coat.	Same procedure.
Boy takes six steps.	You may take six long steps.
I took six long steps.	Tell the class what you did.
Here is the figure six. Here is the word six.	Find the figure six and the word six.
Tin cup and pail.	Pour six cups of water in the pail. Same procedure as above.
Foot rule.	Measure off six feet along the blackboard. Same procedure.
I see a picture of six watches. Here is the word six. Here is the figure six.	Look around the room. If you see six of anything you may tell us about it.
Six boys, six windows, six curtains.	Same procedure.
1—2—3—4—5—6. These six girls may go to their seats.	Count six girls. Tell them to go to their seats.
1—2—3—4—5—6. These six boys may go to their seats.	Count six boys to go to their seats.
There are three boys left.	How many boys are still in class? Run to seats.
There are two girls left in class.	How many girls are left? Run to seats.
	Here are the boxes of pegs. Let us make six green trees on our desks.

A Development of the Idea "Triangle"
Second Grade*

Subject Matter and Material	*Procedure*
Square of pasteboard 10″x 10″.	You have been making borders using different shapes.

*Plan by Miss Grace M. Poorbaugh, Critic Teacher, Second Grade, Bowling Green State Normal College, Bowling Green, Ohio.

Square.

I wonder who can write the name of this shape on the board. (Teacher holds up a square.)

It has four equal sides.

How do you know it is a square?

Oblong of pasteboard 12″x 9″.

Can anyone write the name of this shape on the board? (Teacher holds up an oblong.)

Oblong.

Is is longer than it is wide.

How do you know it is an oblong?

Yes.

Would you like to know another shape that you could use in making borders?

Four triangular pieces of cardboard of different sizes, colors and proportions.

(Teacher sets triangles in chalk rail.)

Does any one know the name of this shape?

Triangle.

No one seems to know it so I'm going to write it on the board for you. (As teacher writes she pronounces the word.)

There are a number of triangles here.

Let us see what we can find out about them.

They are different in size.

Tell one way in which they are different.

They are different in color.

Tell another way in which they are different.

They all have three corners.

See if you can discover one way in which they are all alike.

Let us count the corners and see if Mary is right. (Children count the corners of each triangle.)

They all have three sides.

See if you can discover another way in which they are all alike. What have you discovered?

Let us count the sides and see if Edward is right.

A triangle always has three corners and three sides.

Can you tell us what a triangle always has, Alice?

Pupils suggest: The corners of the blotter pad, the end of a stool, the wigwam in an Indian picture, the iron support on a desk.	Do you see anything in this room that is shaped like a triangle?
Pupils suggest: flat-iron, sister's class pin, end of chicken-coop, and pennants.	There aren't very many things in our room which are this shape. Perhaps you can think of something outside of school.
Books that the pupils have previously made to keep number work in.	Do you think you could draw a triangle? Would you like to try today to make a border using triangles? (Pupils make a decorative border on number books using triangles.)

Inductive Development of Multiplication Table of 3's Second Grade*

Subject Matter and Material	*Procedure*
Yes.	Would you like to play that you are doing some shopping today?
I'd like to go to a Toy Store.	To what kind of a store would you like to go, Mary?
Toys and toy money.	Some of the toys may cost more than we can pay. Suppose we go to the 3 cent counter.
We will need to know the table of 3's.	In order to buy at this counter what will we need to know?
I'd like to buy this paper doll.	Who would like to buy one article?
I will have to pay 3 cents.	How much must you pay for the paper doll, Jane?
I'd like to buy two balloons.	Who would like to buy two articles? Fred may come.
I will have to pay 3 cents + 3 cents or 6 cents.	How much must you pay for two balloons?

*Plan by Miss Grace M. Poorbaugh, Critic Teacher, Second Grade, Bowling Green State Normal College, Bowling Green, Ohio.

I have to take 3 cents two times.

$2\times3¢=6¢$.

How many times do you have to take 3 cents to pay for them?

Can you tell me how to write that on the blackboard?

$1\times3¢=3¢$.

How might Jane's problem be written, Helen?

I'd like to buy three doll chairs.

Who would like to buy three articles? Edna may come.

I will have to pay 9 cents. $3¢+3¢+3¢=9¢$.

How much must you pay for three doll chairs?

I have to take 3 cents three times.

How many times do you have to take 3 cents, Edna?

$3\times3¢=9¢$.

Can you tell me how to write that on the blackboard?

I'd like to buy four marbles.

Would some one like to buy four articles? John may come.

I will have to pay 12 cents. $3¢+3¢+3¢+3¢=12¢$.

How much must you pay for them, John?

I have to take 3 cents four times.

How many times do you have to take 3 cents?

$4\times3¢=12¢$.

Could you tell me how to write that on the blackboard?

(The remainder of the table of 3's is developed in a similar way.)

Postage stamps, post cards, erasers, pencils, marbles, tops, rulers, yeast cakes, 1 dozen Jackstones.

Do you know of any other things you buy at 3 cents each?

I would have to pay $3\times3¢$ or $9¢$.

From what you have learned on the board could you tell what you would have to pay for 3 marbles at 3 cents each, Richard?

I would have to pay $5\times3¢$ or $15¢$.

What would you have to pay for five postage stamps, Harry?

We could learn the table of 3's.

If we wanted to go to the store what could we do to help us, Alice?

Yes.

Do you all think that would help? Then we will try to learn them tomorrow.

Inductive Development of Certain Fraction Facts
Second Grade*

Subject Matter and Material	*Procedure*
	I wonder how many of you ever divide your candy or share your toys with others.
Yes.	Would you like to pretend that this is a play-room and share these things with your classmates?
Candy and toys.	We will let just three children come into the play-room at a time.
I'd like to divide this stick of candy.	Harold, Tom and John may come. What would you like to divide with Harold and Tom, John?
Yes.	Do you think you can divide it equally? You may try.
I have broken it into 3 pieces.	Into how many pieces have you broken the stick of candy, John?
It is one-third.	Do you know what part of the whole stick each of the small pieces is, Tom?
You write one-third this way: ⅓.	Can you write one-third on the blackboard, Harold?
I'd like to divide these 3 blocks.	Bernard, Richard and Earl may come. What would you like to divide with Bernard and Richard, Earl?
We will each have 1 block.	How many blocks will you each have, Earl?
It is one-third.	One is what part of three, Richard?
We have found out that ⅓ of 3=1.	Can you write what you have just found out on the blackboard, Bernard?
I'd like to divide these six paper dolls.	Mary, Helen and Alice may come. What would you like to divide with Mary and Helen, Alice?
We each have 2 paper dolls.	How many paper dolls do you each have, Alice?

*Plan by Miss Grace M. Poorbaugh, Critic Teacher, Second Grade, Bowling Green State Normal College, Bowling Green, Ohio.

It is one-third.

Two is what part of six, Mary?

We have found out that ⅓ of 6=2.

Can you write what you have just found out on the blackboard, Helen?

I'd like to divide these 9 crackers.

Jane, Edna and Emma may come. What would you like to divide with Jane and Edna, Emma?

We each got 3 crackers.

How many crackers did you each get, Jane?

It is one-third.

Three is what part of nine, Edna?

We have found out that ⅓ of 9=3.

Can you write on the blackboard what you have found out, Jane?

I'd like to divide these marbles.

Frank, Fred and James may come. What would you like to divide with Frank and Fred, James?

We each have 4 marbles

You may try.

How many marbles do each of you have, James?

We have found out that ⅓ of 12=4.

Can you write what you have found out on the blackboard, Frank?

(In a similar way ⅓ of 15, 18, 21, 24, 27, 30, 33 and 36 are developed.)

Pupils suggest apples, oranges, cookies, candy beans, and chocolates.

Do you ever divide other things into thirds?

Each child would get 4 candy beans.

From what we have written on the blackboard can you tell how many candy beans 3 children would get if 12 beans were divided equally among them, Frank?

Each child would get 2 cookies.

If 6 cookies were divided equally among 3 children how many would each get, Edna?

Each child would get 6 chocolates.

If 18 chocolates were divided equally among 3 children, how many would each get, Helen?

We could learn what we have on the blackboard.

In order to be able to divide things with others what could we do to help us, Fred?

Yes.

Would you like to do that?

Then we will try to learn these tomorrow.

A Deductive Development of the Multiplication Table of 9's

Third Grade*

Subject Matter

Method of Procedure

How many like to play baseball? We have on the board a game called "Making a Home Run." Look at the figure in the center of the diamond. To play this game and make a home run, we have to multiply 9 by the numbers at each corner.

We shall need to know the table of 9's.

What shall we need to know in order to make a score?

$1\times9=9$.

Who can give the first combination?

$2\times9=18$.

Give the next one.

No, the product is just the same.

Does it make any difference in the product whether we say 3×9 or 9×3?

$9\times3=27$.

How many are 9×3? Then how many are 3×9?

$3\times9=27$.

We know them to 9×9.

$4\times9=36$ $7\times9=63$

$5\times9=45$ $8\times9=72$

$6\times9=54$

Using what we have learned in our other tables, how far do we really know the table of 9's? I will write the combinations and you give the product.

Add 9 to 72.

If eight 9's are 72, how could you find nine 9's?

$9\times9=81$.

Mary, tell us the product of 9×9.

$10\times9=90$.

How many are 10×9?

(11×9 and 12×9 are developed in the same way.)

---

*Plan by Miss Effie Alexander, Critic Teacher, Third Grade, Bowling Green State Normal College, Bowling Green, Ohio.

On the last part beginning with 9×9.

On what part of the table do we need to put the most study?

Study 9×9 and 10×9; look away from the board and say them. I will erase the answers. Write the combinations on your papers. Study the last two combinations. I will erase the answers and you may write them on your papers. Study.

Yes.

Are you ready to play the game?

1×9= 9
5×9=45
9×9=81

Who can make first base?

10×9=90.
6×9=54.
4×9=36
etc.

Who can get from first base to second?

Tom, run from second to third.

Now, Harold try to get in home.

Who thinks he could make a home run?

(Several children make a home run.)

6×9 8×9
7×9 9×9

Some of us are having trouble making our bases. Which combinations are giving us trouble?

Practice the table of 9's.

What would you suggest that we do tomorrow?

A Development of the Process of Carrying in Addition Third Grade*

Subject Matter	*Method of Procedure*
Scores Boys 12 Girls 16	The score of the boys yesterday was 12 and that of the girls was 16.
We will add the scores made today to those we had yesterday.	How shall we find what the scores are now that the boys have made 16 today and the girls 15?
12 16 28	Marion may write the scores of the boys and add them.

*Plan by Miss Effie Alexander, Critic Teacher, Third Grade, Bowling Green State Normal College, Bowling Green, Ohio.

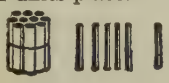
16
15

No.

12
16

The boys' score was 12, which equals 1 ten and 2 units. I will place 1 bundle of 10 sticks for the 1 ten and 2 sticks for the 2 units. 16 equals 1 ten and 6 units. I will place one bundle of 10 sticks under the other tens and 6 sticks for the 6 units under the two sticks and add. 6 units plus 2 units equal 8 units. I will place 8 sticks in units place. 1 ten plus 1 ten equals 2 tens. I will place two bundles of ten sticks in tens column. The sum is 2 tens and 8 units, which equals 28.

Place one bundle of ten sticks in tens place and 6 sticks in units place.



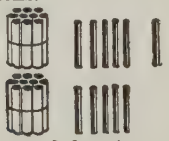
Garness may add the scores of the girls. (Pupil could not add.)

Can any one in the class help her? Then, we shall have to learn a new process in arithmetic which we call carrying in addition.

Let us use the sticks and find the boys' score. Alden may do that telling us what he thinks as he does it.

Keeping in mind what Alden has just done, let us find the girls' score by using the sticks. How shall I represent 16, the score that the girls had yesterday?

Place one bundle of 10 sticks in tens place and 5 sticks in units column under the 6 sticks.



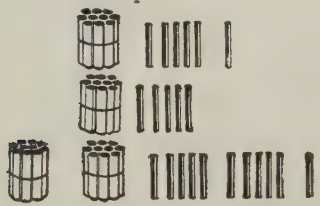
What shall I do with the score that was made today?

5 units and 6 units are 11 units. Place 11 sticks in units place.

We are now ready to add. What shall I place in unit's column?

1 ten and 1 ten equals 2 tens. Place two bundles of sticks in tens place.

What shall I place in ten's column?



There are 11 sticks.

Let us look at the number of sticks in units place. How many are there?

11 equals 1 ten and 1 unit.

How many tens and units does 11 equal?

Yes, put 10 sticks into a bundle which equals 1 ten and place the other stick to the right of the bundle.

Could we show that with the sticks?

1 unit.

What do we call the stick that is left?

In ten's column.

Where should we put the tens?

I will place the one ten with the other tens in ten's column and leave the 1 stick in unit's column.

We have 3 tens and 1 unit. It equals 31.

What have we as our sum now?

What number does that equal?

15
19

Helen may add 15 and 19 using the sticks and telling us what she thinks. (Same procedure.)

17 18	Who could add 17 and 18 without using the sticks and telling us what he thinks?
8 units and 7 units are 15 units or 1 ten and 5 units, write the 5 units and add the 1 ten to the other tens. 1 ten and 1 ten and 1 ten are 3 tens, write the tens.	
13 29	Is that clear to everyone? If so, Lenore may add 13 and 29 using as few words as possible.
9, 12 write 2, carry 1. 3, 4 write 4. The sum is 42.	
15 26 34 38 27 18	Same procedure.
15 18 27 14 17 39 29 49	Each one may add these numbers on paper.
We have learned to carry in addition.	What have we learned in today's lesson?
We should drill on carrying in addition.	What do you think that we should do in tomorrow's lesson?

A Development of the Process of Uneven Division Third Grade*

Subject Matter	*Method of Procedure*
Bean Bag Game Scores Reds 65 Blues 78	How do our scores stand at the end of the game?
The Blues.	Which is ahead?
But that isn't fair because the Reds have only 5 on their team while the Blues have 6 to make their score.	

*Plan by Miss Effie Alexander, Critic Teacher, Third Grade, Bowling Green State Normal College, Bowling Green, Ohio.

No.

With 11 in the class, could we make a fairer division of the number?

Find the average number done by the members of each team.

What shall we do in order to make a fair comparison of the scores?

Divide the score of the Reeds by 5 and the score of the Blues by 6.

How can that be done?

Short division.

What process will you use?

$5\overline{)65}$

Emerson may get the average score for his team. Place your example on the board.

(Pupil cannot solve this example as he does not know what to do with the remainder that arises in dividing 6 by 5.)

We will learn how to do this kind of examples tomorrow, so that you will have no trouble in getting your average score.

Next day's lesson.

Subject Matter

We were to learn how to divide 65 by 5 and 78 by 6 in order to find which team has the highest average score.

Method of Procedure

What was our problem for today?

Let us recall some of the things that we know about numbers and about division.

48 equals 4 tens and 8 units.

How many tens and units does 48 equal? 36? 96?

36 equals 3 tens and 6 units.

96 equals 9 tens and 6 units.

$3\overline{)96}$

How many 3's are there in 9? 3. Write the 3 above the 9. How many 3's are there in 6? 2. Write the 2 above the 6. The quotient is 32.

Fred may divide 96 by 3 telling us what he thinks and does.

$5\overline{)65}$

65=6 tens and 5 units.

Now, let us try to divide 65 by 5. How many tens and units are there in 65?

There is one 5 with one remainder.

How many 5's are there in 6?

Place the 1 above the six.

Where shall we place this part of the quotient?

It is one ten.

What is the 1 remainder?

It equals 10 units.

How many units does 1 ten equal?

Add them to the other units.

Can any one suggest what we might do with these ten units?

10 units+5 units=15 units.

How many units will that give us?

Divide 15 units by 5.

What will the next step be?

How many 5's in 15? There are 3. Write the 3 above the 5. The quotient is 13.

What do we think and do?

Yes.

Is that clear to everyone?

$6\overline{)78}$

How many 6's in 7? One with 1 remainder. Write the 1 above the 7. The 1 ten remainder equals 10 units. 10 units and 8 units equal 18 units. How many 6's in 18? 3. Write the 3 above the 8. The quotient is 13.

Then, Marion may find the average score for the Blues telling us what he thinks and does.

Neither. It is a tie.

Which team has the highest average score?

Let us work some more of these examples so that we shall have no trouble in the future.

$4\overline{)56}$	Eleanor may divide 56 by 4, using as few words as possible.
4 in 5, 1 and 1 over; 4 in 16, 4.	(The form given under Subject-Matter is the one finally decided on.)
$3\overline{)45}$ $2\overline{)56}$ $4\overline{)52}$	(These are handled in the same way. The work is individual and oral.)
$5\overline{)65}$ $6\overline{)72}$ $4\overline{)56}$	
Yes.	Is every one ready to work these examples in that way?
$7\overline{)84}$ $4\overline{)96}$ $3\overline{)72}$	You may work these examples on paper.
$3\overline{)75}$ $2\overline{)38}$ $2\overline{)94}$	

A Development of Rule for Checking Division
Third Grade*

Subject Matter	*Method of Procedure*
Yes.	We have learned to test our answers in addition, subtraction and multiplication. Would you like to feel certain that the quotient which you get by division is correct?
	All right we will learn to check our answers in division.
	Let us work these examples.
$3\overline{)12}$	If we divide 12 by 3 what is the quotient?
The quotient is 4.	
$4\overline{)28}$	If we divide 28 by 4 what is the quotient?
The quotient is 7.	
$9\overline{)54}$	What is the quotient when we divide 54 by 9?
The quotient is 6.	
$7\overline{)63}$	When we divide 63 by 7?
The quotient is 9.	
4 7 6 9 $3\overline{)12}$ $4\overline{)28}$ $9\overline{)54}$ $7\overline{)63}$	Let us examine the first example that we have worked.
How many 3's in 12? There are 4.	What did we think when we did the first example?
Yes.	Is that true? If there are 4 threes in 12, then 4×3 must equal 12. Is that true?

*Plan by Miss Effie Alexander, Critic Teacher, Bowling Green State Normal College, Bowling Green, Ohio.

Yes.

Let us examine the other examples. Does $7\times4=28$; $9\times6=54$, and $7\times9=63$?

We found that if we multiplied the quotient and divisor together the product was equal to the dividend.

What did we find true in each case?

234	321	324
2)468	3)963	2)648

Let us try this with examples that need to be worked with pencil.

Agnes may take the first, Pansy the second, and Neva the third.

234	321	324
2	3	2
468	963	648

Fred may see if the product of the quotient and the divisor equals the dividend in Agnes' example, Herman may take Pansy's, and Helen take Neva's.

Yes.

Did it prove true in each one of these?

To check division multiply the quotient and the divisor. If the product equals the dividend the quotient is correct.

I wonder who is ready to tell us how he could be sure that his quotient was correct. Eleanor may.

3)933	4)484	7)147
6)846	3)693	8)968

We will work and test these examples.

A Development of the Process of Subtracting Mixed Numbers

Fifth Grade*

Subject Matter

Method of Procedure

When we finished our lesson yesterday, I told you there was just one more step in the subtraction of mixed numbers, that we had to learn. Your papers yesterday, reviewing the steps we have learned so far, were good, so I think we are ready to go on with the new work today. Let us first

*Plan by Miss Minnie Ullrich, Critic Teacher, Fifth Grade, State Normal College, Bowling Green, Ohio.

work each kind of example we have had, so we will be sure to have the steps in mind. They may help us to determine what to do in our new work.

I. Review, in preparation for new work.

A. 1. $8\frac{5}{6}-3\frac{1}{6}=?$

2. $85\frac{15}{16}-38\frac{7}{16}=?$

You may pass to the board. Number by 2's. Number 1's may take the first example I read, number 2's, the second. (In all this review work, the teacher and class will note any errors or difficulties that may arise, stopping to make the necessary corrections and suggestions. Such things, for example, as arrangement of work, or reduction of the answer, may call for criticism.)

B. 1. $72\frac{5}{6}-48\frac{1}{4}=?$

2. $142\frac{2}{3}-40\frac{1}{4}=?$

What did you have to do in this example that you did not do in the other one?

Reduce the fractions to their least common denominator.

C. 1. $11-7\frac{5}{7}=?$

2. $20-8\frac{3}{16}=?$

What was the new step in this subtraction?

Changing the minuend to a mixed number.

II. The new process: Subtracting mixed numbers, when a change in the minuend is necessary.

Example: $14\frac{2}{3}$

$-9\frac{3}{4}$

(Place the example on the board.)

First step: Change the fractions to the least common denominator.

In order to subtract $9\frac{3}{4}$ from $14\frac{2}{3}$, what is the first thing we must do?

$14\frac{2}{3}=14\frac{8}{12}$

$9\frac{3}{4}=9\frac{9}{12}$

(The teacher puts the example on the board, writing answers from the pupils' dictation.)

Second step: Subtract the fractions.

In the examples we *have* worked, what did we do after reducing the fractions to the least common denominator?

Difficulty: We cannot subtract $\frac{9}{12}$ from $\frac{8}{12}$.

Very well, let us subtract the fractions. What difficulty do you find? Can anyone suggest something to do? When we subtracted a mixed number from an integer, as in this example, (referring to one previously worked at the board; $20-8\frac{3}{16}$) and found we had nothing to subtract the fraction from, what did we have to do? Don't you think it might help us to do that here?

New step: Change the minuend.

$13\frac{12}{12}$.

To what whole number and fraction would you change 14?

$13\frac{12}{12}+\frac{8}{12}$.

What have we now, instead of $14\frac{8}{12}$?

$13\frac{20}{12}$.

$14\frac{2}{3}=14\frac{8}{12}=13\frac{20}{12}$.

$9\frac{3}{4}=\ 9\frac{9}{12}=\ 9\frac{9}{12}$.

That makes 13 and how many 12ths?

$\frac{9}{12}$ from $\frac{20}{12}=\frac{11}{12}$.

9 from $13=4$.

Completed form:

$$\begin{array}{r} 14\frac{2}{3}=14\frac{8}{12}=13\frac{20}{12} \\ 9\frac{3}{4}=\ 9\frac{9}{12}=\underline{\ 9\frac{9}{12}} \\ 4\frac{11}{12} \end{array}$$

Now can we subtract the fractions? You may finish the example, and I'll write your answer.

Read the answer. What was the only *new* thing we did in this example? (Changing the minuend.)

IV. Drill.

$15\frac{1}{6}-8\frac{1}{3}=?$

Do you think you can work one now? Let us work one more together. I'll write it on the board. See if you can tell me just what to do, without my helping you. (Call on various individuals for the different steps.)

$20\frac{1}{4}-2\frac{5}{8}=?$

$15\frac{2}{5}-4\frac{7}{10}=?$

$75\frac{3}{8}-44\frac{2}{3}=?$

The class may pass to the board, and we'll see if each of you can do this kind of subtraction example now. I wonder if any one will get all of them right the first time we try them.

NOTE: As soon as the pupils have mastered the process in its complete

form the work is abbreviated, the final form being:

$14\frac{2}{3}$	12
$9\frac{3}{4}$	8
$4\,11\frac{1}{12}$	9
	$11\frac{1}{12}$

The change in the minuend is made mentally.

Drill Lesson on Addition Combinations
First Grade*

Subject Matter

Yes.

3	4	5	6	7
4	4	4	2	0
5	7	8	8	5
3	2	0	1	2
9	7	6	6	
0	1	3	1	

Work on these numbers.

We know	6	7	8			
	1	1	1			
4	7	6	8			
4	0	0	0			
3	3	5	2	7	6	
4	6	4	6	2	2	
7	9	9	8	9	8	
5	4	6	4	6	3	2
3	5	2	3	3	5	7
8	9	8	7	9	8	9

Procedure

Last week we played Fish Pond with the fives and sixes. Would you like to play again today?

Today we will use the sevens, eights and nines so we can catch bigger fish.

Who can give me two numbers which make seven? (Same question for eight and nine. Teacher writes combinations on board as they are given.)

I wonder if we are ready to play the game? Can any one give these numbers without making a mistake? (First two pupils hesitate and make mistake.)

What had we better do before we start catching fish?

Look at these numbers carefully. Are there any that you know so well you do not need to study them?

Look at the hard ones that are left.

Who is sure that he knows the answer to the first? (Same procedure with the others. Teacher writes results on board as given.)

Now we will study these and when we know them we will play Fish Pond. Study the first four.

*Plan by Miss Lucy H. Meacham, Critic Teacher, First Grade, Bowling Green State Normal College, Bowling Green, Ohio.

John, you may give them. (Teacher erases answers. Other pupils give the first four and then the remaining combinations are dealt with in the same way.)

Yes.

Do you think you can play Fish Pond now?

5	7	8	8	5	3	4	3
3	1	0	1	2	4	4	5
6	5	7	7	6	9	6	1
2	4	2	0	3	0	1	7
0	1	2	4	2	4	2	0
8	8	5	3	6	5	7	7
3	0	1					
6	9	6					

(The teacher writes all of the combinations that make 7, 8 and 9 on the board in miscellaneous order.)

We will draw a line around these numbers and play this is a fish pond.

Five and four are nine.

Who would like to go fishing first? John.

(As the fish is caught by giving the sum correctly the pupil erases the combination. This is continued until all of the fish are caught. If a sum is given incorrectly the pupils play that the fish got off the hook and slipped back into the pond. The pupils try to catch big fish (difficult combinations) instead of little minnows.

6 7
3 2 etc.

Before we go to our seats tell me one combination you are sure of. (Each pupil gives one combination and takes his seat.)

Yes.

Would you like to play Fish Pond and other games again some day?

Drill on the Roman Numerals
Second Grade*

Subject Matter	*Procedure*
	A lot of new books have come for our grade. These books are different from all of our other books. The whole book is one long story and the story is divided into chapters. (Teacher shows one of the books to the class.)
Yes.	Do you see how these chapters are numbered?
We will need to know the Roman numerals.	In order to find the chapters in these books what will we need to know?
They are on the clock.	Where else are numbers like these used?
Material: Educational clock dial.	Would you like to play a clock game to see how well you know them? (Teacher holds up clock dial.) Who can show us on the clock the time you get up in the morning?
I get up at six o'clock.	Emma may set the hands and tell the class. (The time the pupils eat breakfast, start for school, have recess, and go home at noon are handled in the same way. Several pupils do not know the Roman Numerals very well and have trouble finding them on the clock.)
We don't know these numbers very well.	You don't seem to be able to set the hands quickly. What is the trouble?
Drill on them.	What had we better do? Let us try to get them so well learned that we won't have trouble again.

*Plan by Miss Grace M. Poorbaugh, Critic Teacher, Second Grade, Bowling Green State Normal College, Bowling Green, Ohio.

I is 1
II is 2
III is 3

Who can recite the first three for us?

(Teacher writes them on the board as they are given.)

IV is 4
V is 5
VI is 6

Who can recite the next three?

VII is 7
VIII is 8
IX is 9

Who can recite the next three?

X is 10
XI is 11
XII is 12

Who wants to recite the last three?

Pupils suggest and teacher writes on board:

IV is 4
VI is 6
IX is 9
XI is 11

Which do you find are the hardest for you?

I before V means we subtract 1 from 5 and

$5-1=4$

I after V means we add 1 to 5 and

$5+1=6$

I before X means we subtract 1 from 10 and

$10-1=9$

I after X means we add 1 to 10 and

$10+1=11$

Can you tell us something that will help us remember these?

All study these four.

IV is 4
VI is 6
IX is 9
XI is 11

Who would like to recite these four most difficult ones? (Several pupils recite.)

11 5 3 4 2 6 8 7 10 9 12
7 12 2 5 9 8 4 11 10 6 3

Let us try this game. (Two rows of numbers are written on the board.) The front row may race with the back row to see who can write the Roman numerals under these first.

May and Lester may come first.

The front row won.

IV is	IX is
IX is	V is
VII is	IV is
VI is	VIII is
XI is	X is
VIII is	VII is
X is	XI is
V is	VI is

Which row won, Lois?

Let us have another race and see if the back row can get them more quickly. The first row may race with the back row again. This time you may see who can write the answers first in these columns. Emma and Harry may come. Dorothy may keep the score. The rest will watch to see that the score keeper gives the score to the right side.

The back row won.

Which row won, Dorothy?

Let us try one more game to see if we know the four most difficult ones.

Is it IV is 4?

I'm thinking of one of the four. Can you guess it?

No, it is not.

Guess again.

Is it XI is 11?

No, it is not.

Guess again.

Is it VI is 6?

No, it is not.

Guess again.

Is it IX is 9?

Yes, it is.

Yes.

Do you think you could tell time more quickly now?

Let us try the clock game again.

I eat my dinner at 12 o'clock.

Who can show us the time you eat your dinner?

(The time school begins and closes in the afternoon, and the time the pupils eat supper and go to bed are handled in the same way.)

Yes.

Do you feel that you know the Roman Numerals any better?

Let us see if you can find the chapters in our new books now.

Find Chapter IX.

(The pupils find the first twelve chapters.)

Drill on Subtraction Combinations
Third Grade*

Subject Matter

Method of Procedure

In the speed test on subtraction that you took yesterday you did not make many mistakes but most of you did not finish as many examples as you should. Do you know what the trouble is?

Let us see if we can find out.

966 973
349 248

824 643
162 287

Clifton may solve the first problem, telling us just what he thinks as he does it. Helen may take the next one, etc.

We are slow in giving the combinations.

We are not having much trouble with the way in which the problems are solved but I think that I have discovered our trouble. Do you see what it is, Harold?

Drill on giving the combinations rapidly.

Those making:

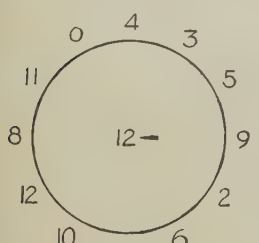
13 15 17
14 16 12

What shall we do to improve?

Do you know which ones trouble you most? (Different members of class give those with which they have trouble.)

12 12 12 12 12 12 12
−3 −8 −7 −5 −6 −9 −4

Two or three children give the results as rapidly as possible.



Two or three children go around the circle as rapidly as possible.

Drill on the combinations making 13, 14, 15, 16 and 17 in a similar manner.

Papers on which the combinations are printed are passed and a 1 minute test for speed and accuracy is held.

Do you feel that you are better prepared to take up our work in subtraction now? We will take that for our lesson tomorrow.

*Plan by Miss Effie Alexander, Critic Teacher, Third Grade, Bowling Green State Normal College, Bowling Green, Ohio.

Drill: Changing Fractions to Higher Terms
Fifth Grade*

Subject Matter

Procedure

Yesterday we learned how to change fractions to higher terms. We all understand how to do the work, but we haven't had practice enough to do it quickly and easily. We shall have to use this process very often as we go on with our work in fractions, and if we can't do it readily, we'll be hampered in our new work. You have been anxious to learn how to add fractions. Just as soon as you can change fractions to higher terms, you'll be ready for the addition. So let's see if we can't get enough practice today to help each one to do the work quickly and correctly.

How many can tell how to change a fraction to higher terms?

To change a fraction to higher terms multiply both the numerator and the denominator by the same number.

What principle did we discover a few days ago, that gives us the right to do this?

Multiplying or dividing both terms of a fraction by the same number does not change the value of the fraction.

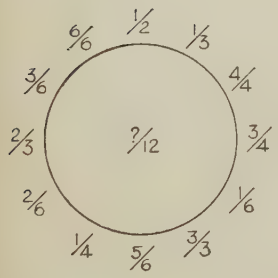
1. $\frac{1}{2}=?/4$
   $\frac{1}{3}=?/6$
   $\frac{2}{3}=?/6$
   $\frac{1}{2}=?/8$
   $\frac{1}{4}=?/8$
   $\frac{3}{4}=?/8$

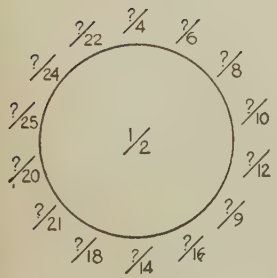
We'll take a few very easy ones first, in order to be sure that everyone remembers just how to do them. You may tell us by what number you will multiply the numerator and denominator, and then read your example with its answer.

(Different pupils take one at a time.)

*Plan by Miss Minnie Ullrich, Critic Teacher, Fifth Grade, Bowling Green State Normal College, Bowling Green, Ohio.

2. $\frac{1}{3}$=?/ 9=?/12=?/15
$\frac{2}{3}$=?/ 6=?/ 9=?/12
$\frac{3}{4}$=?/ 8=?/12=?/16
$\frac{2}{5}$=?/10=?/15=?/20
$\frac{1}{7}$=?/14=?/21=?/28





Read these statements, supplying the missing numbers.

Those who made mistakes in any of these, stand. Have you any question on them now? Take the ones you had trouble with, once more.

Change the fractions outside of the circle to 12ths. We'll begin with Robert and go right around the class. If anyone makes a mistake, the next person be sure to correct it. Change the fraction to which I point. (Point to the fractions first in rotation, then out of order.)

Now I'd like several people to begin and go entirely around the circle without making a mistake. Who is ready to try? (Call on several.)

(Change ?/12 in center of circle to ?/24.) Robert, you may take the pointer and call on whomever you wish to recite. You may ask the class to work in any way you wish.

Change the fraction in the center to as many of the denominators on the outside, as possible. Beginning with 4ths, go around the circle toward the right. (Change $\frac{1}{2}$ to $\frac{1}{3}$, $\frac{3}{4}$, etc.)

In your books is a list of fractions to be changed to higher terms. You may write the answers only, and we'll see how many can write every answer correctly in one minute. Open books to page 44. The list is at the top of the page. Ready. Begin— (Time the class.) Time. You may check mistakes as I read answers. How many had them all right? Those who made mistakes, stand. Read the ones you missed, correcting the mistakes.

Tomorrow we shall learn how to add fractions.

Drill Lesson

Sixth Grade*

Subject Matter	*Method of Procedure*
Most of the scores were below the median.	How did your scores in the Courtis Test compare with the class median given you?
More drill is needed upon the fundamentals.	The last time the Studebaker Test was taken, only a few finished the cards. What does this indicate? Before taking the tests today, let us spend some time drilling.
Flash cards for addition. 9 6 4 7 5 3 etc.	Give answers quickly and quietly to the addition combinations as they are held up. Class correct mistakes. 1st. In concert. 2nd. By rows. 3rd. Individuals. 4th. Boys. 5th. Girls. 6th. One pupil who needs extra drill.
6 7 8 4 2	Give answers to the column of figures placed upon board and then quickly erased.
40316498432 −29489569871	Let us drill upon subtraction. J—— may subtract first.
10826928561 −2583496874 etc.	(Other individuals called upon until all those needing drill are called upon.)
Card held up. Cards are numbered from 0 to 12 but are not arranged consecutively.	Multiplication drill. Give answers as card is held up. Multiply by 2. Multiply by 9. Multiply by 7, etc.

*Adapted from a plan by Miss Ella J. Holley, Critic Teacher, Sixth Grade, Bowling Green State Normal College, Bowling Green, Ohio.

Vary by calling upon different rows, sections, and individuals as was done for drill in addition.

Numbers from 3 to 89 written on board.
3 4 5 6 7 8 9 10 11 12 13
14 15 16 17 18 19 20 21 22
23, etc.

Teacher points to numbers at random. Divide by 2, by 7, etc. Pupils give quotient and remainder each time. Vary by calling on different groups and individuals as in addition and multiplication.

Studebaker Tests.

Problem Recitation—Planning a Party

Second Grade*

Subject Matter	*Procedure*
Yes.	Do you like to go to parties? Would you like to play that you were planning a party?
You must decide the number of children you will invite.	What is the first thing to decide when you have a party?
There will be 12 children invited. $6+6=12$	If you decide to invite 6 girls and 6 boys how many children will be invited?
The invitations will cost 24¢. $12\times 2¢=24¢$	If you get invitations which cost 2¢ each how much will they cost?
I think the time for the party should be decided.	What is the next thing to be decided?
I nearly always go at 2 o'clock.	What time are you usually invited to go to parties?
Until 5 o'clock.	How long do you usually stay?
Yes.	Would you like to have this party from 2 to 5 o'clock.
Clock face.	Who can set the clock for 2 o'clock? What can set it for 5 o'clock?

*Plan by Miss Grace M. Poorbaugh, Critic Teacher, Second Grade, Bowling Green State Normal College, Bowling Green, Ohio.

There will be 9 children at the party.

$12-3=9$

If three of the children cannot come to the party how many will be there?

We should decide what we will serve for refreshments.

We have decided the number we will invite and have the invitations. What must we decide next?

Pupils suggest: Ice cream, cake, sandwiches, pie and candy.

What do you like to have to eat when you go to parties?

No.

Do you think we should have all of these things?

I think sandwiches and ice cream and cake would be nice.

Which do you think we had better choose?

Yes.

Do you all think that would be nice?

We must make 27 sandwiches.

$9\times3=27$

If we make enough sandwiches so each child may have 3 how many must we make?

We must have 18 pieces.

$9\times2=18$

If we have enough cake so each child may have 2 pieces how many pieces must we have?

We will have 10 pints.

1 quart $=2$ pints

5 quarts $=5\times2$ pints $=$ 10 pints

If we buy 5 quarts of ice cream how many pints will that be?

I think it would be nice to have little chairs and tables.

We have decided what we will have to eat. Is there anything else you would like to have?

We will need 3 tables.

$9\div3=3$

If we seat 3 children at each table how many tables will we need to order?

Yes.

Would you like to decorate the tables?

I think flowers and candles would be pretty.

What do you think would be pretty on them?

We must have a dozen flowers.

$3\times4=12$

If we put a vase with 4 flowers on each table how many flowers would we need?

We will need 6 candles.
$3\times2=6$

If we put 2 candles on each table how many candles will we need?

They will cost 30 cents.
$6\times5¢=30¢$

If the candles cost 5 cents each how much will they all cost?

.30
$1.25
.60
$2.15

If the candles cost 30 cents, the ice cream $1.25, and the cake 60 cents, can you find out the total cost of these three things?

Show us on the board.

I like to play games.

What do you like to do at parties?

There will be three groups.
$9\div3=3$

If you play a game in which the children are divided into groups of 3 how many groups will there be?

Three of the children go.
$\frac{1}{3}$ of $9=3$

If one-third of the children start home together how many go?

There are 6 children left.
$9-3=6$

How many are left?

Three children go.
$\frac{1}{2}$ of $6=3$

If one-half of them go next how many go?

There are 3 left.
$6-3=3$

How many are left then?

She lives 11 blocks.
$8+3=11$.

Two of the children live 8 blocks away; the other one lives 3 blocks further. How far does she live?

Problem Recitation

Third Grade*

Subject Matter

As a language lesson the pupils had each handed in the previous day a list of three original problems. The girls' problems were about buying doll clothes, the boys' about buying toys and games.

Procedure

Look at your problems and choose one that you want the class to solve.

---

*Plan based on a lesson given by Miss Effie Alexander, Critic Teacher, Third Grade, State Normal College, Bowling Green, Ohio.

I went to the store to buy some silk for my doll. I paid $1.50 a yard and I bought 2 yards. How much did it cost?

Goldie, you may read yours.

The cost of one yard of silk and the number of yards.

What does the problem tell you, Harry?

The total cost of the silk.

What does it ask?

Multiplication.

What process will you use?

$3.

What is the answer?

I bought a Tinker Toy for $.80, an animal game for $.15 and a windmill for $.30. How much change did I get from $2?

John, read one of your problems.

The cost of each toy and how much money John gave the clerk.

What does the problem tell, Alice?

How much change he got.

What does it ask?

How much the toys cost.

What must you find first?

Add.

What will you do to find out?

Come to the board and do it for us.

$.80
.15
.30
$1.25

How much change John received.

What can you find next?

Subtraction.

What process will you use?

$.75.

How much change did he get?

Yes.

Is that correct, John?

I bought 6 doll dresses for $.25 each and 2 pairs of shoes at $.75 each How much did I spend? Etc.

Same procedure. The answer was obtained mentally whenever possible.

Problem Recitation—Commercial Discount
Sixth Grade*

Subject Matter	*Procedure*
Pupils have cut from local newspapers advertisements of sales, especially those mentioning per cent.	Read one of your advertisements.
Pupils suggest— Merchants want to dispose of odd pieces, and goods that are going out of style or season, to encourage buying in dull seasons, to bring new customers to the store, etc.	Why are these put in the papers?
Advertisement. 25% discount on any chair in this special display.	If you should go to that store and find a chair marked $20, for how much less than the marked priced would you get it?
¼ of $20=$5	
$20−$5=$15	What would you have to pay for the chair?
Definition of terms: (a) Discount. (b) Rate of discount. (c) Marked price. (d) Net price.	
Other similar problems based on advertisements.	Same procedure. The terms "discount," "net price," etc., are used. For tomorrow make up and solve five problems from data given in advertisements.

*Adapted from a plan by Miss Ella J. Holley, Critic Teacher, Sixth Grade, Bowling Green State Normal College, Bowling Green, Ohio.

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Overman, J. R.

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